Equilibrium shapes of a heterogeneous bubble in an electric field: a variational formulation and numerical verifications

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The equilibrium shape of a bubble/droplet in an electric field is important for electrowetting over dielectrics (EWOD), electrohydrodynamic (EHD) enhancement for heat transfer and electro-deformation of a single biological cell among others. In this work, we develop a general variational formulation in account of electro-mechanical couplings. In the context of EHD, we identify the free energy functional and the associated energy minimization problem that determines the equilibrium shape of a bubble in an electric field. Based on this variational formulation, we implement a fixed mesh level-set gradient method for computing the equilibrium shapes. This numerical scheme is efficient and validated by comparing with analytical solutions at the absence of electric field and experimental results at the presence of electric field. We also present simulation results for zero gravity which will be useful for space applications. The variational formulation and numerical scheme are anticipated to have broad applications in areas of EWOD, EHD and electro-deformation in biomechanics.

1. Introduction

Electro-mechanical coupling is of fundamental importance for many applications. For solid and hard materials, electro-mechanical coupling manifests itself as
piezoelectric or flexoelectric effects [1]. For soft materials and fluids, electro-mechanical coupling is often addressed by invoking the concept of Maxwell stress [2, 3]:

\[
T_{MW} = E \otimes D - \frac{\epsilon(x)}{2}|E|^2 I,
\]  

(1.1)

where \( E \) (\( D \)) is the local electric field (displacement), \( I \) is the identity tensor and \( \epsilon(x) \) is the local electric permittivity. Though widely used for many applications [4, 5], the fundamental concept of Maxwell stress is not free of contention [6, 7]. In the literature, the following quantity (e.g. [8, ch. 3, 9])

\[
T_0 = E \otimes D - \frac{\epsilon_0}{2}|E|^2 I
\]  

(1.2)

is also referred to as the Maxwell stress with \( \epsilon_0 \) being the vacuum permittivity. Moreover, the physical interpretation of the Maxwell stress often relies on the formula that the electric body force is given by \( f = \text{div} T_{MW} \) which, by direct calculations, yields body force terms called electrophoretic force, dielectrophoretic force, electrostrictive force, etc. [10, 11]. It is somewhat mysterious what are the distinctions and physical origins of the above different formulae of Maxwell stress and various forces due to electro-mechanical interaction. This conceptual difficulty is not new and has been discussed in length in the literature [2, p. 132, 8, ch. 3, 12]. In addition, for static equilibrium problems, we anticipate the thermodynamic state of the system shall be such that the free energy of the system is minimized. This motivates us to establish a free-energy based variational formulation for continuum bodies that accounts for general electro-mechanical couplings [13, 14].

In this work, we consider a particular class of electro-mechanical coupling problem: the equilibrium shape of an immersible bubble in an electric field. The field equation for the equilibrium interface is well known, i.e. the generalized Young–Laplace equation with a force term contributed by the Maxwell stress [15, 16]. One may wonder if the field equation admits a variational formulation, and if so, the formulation shall give a precise explanation on the origin of the Maxwell stress.

We remark that the problem of equilibrium shape and evolution of a bubble arises from a number of important applications including electrowetting over dielectrics (EWOD), electrohydrodynamic (EHD) enhancement for heat transfer [17, 18], and electro-deformation of a single biological cell [19]. Similar problems also arise in the determination of the equilibrium shape of a pore channel embedded in a soft elastomer matrix coupled with electrostatics and diffusion [20–22]. To demonstrate our formulation, we focus on a single vapour bubble model for EHD as it represents a simple yet relevant model of EHD-enhanced boiling [10, 15, 16, 23–28]. Experimental observations have confirmed that the electric field can alter the bubble dynamics dramatically, for instance, resulting in elongated bubble shape along the field direction, increased ebullition frequency and smaller departure volume [26, 29–32]. Meanwhile, a few theoretical and computational studies have been conducted to understand the physical mechanisms for the effect of electric field on the behaviour of EHD bubbles [15, 28, 33–38]. However, to the authors’ best knowledge, the existing theoretical treatments and numerical implementations are based on the field equations where the expression of Maxwell stress (1.1) is taken for granted. Moreover, the stability criterion for bubbles remains elusive in the field equation formulation, demanding a more systematic energetic analysis.

The energy-based approach in EHD has been employed by Cheng & Chaddock [39] to analyse the deformation and stability of spheroidal homogeneous bubbles in an electric field where the effect of the solid substrate is neglected. In the subsequent work [30], the authors obtained the shape profile of a bubble elongating in the electric field and determined the departure size at different electric field strength. Cheng & Chaddock’s [30, 39] approach was based on the assumption of ellipsoidal shapes and the closed-form solutions to the electrostatic problem where the concept of (Maxwell) stress was not needed at all, and hence cannot be directly used for numerical simulations. Our work extends the energy-based approach to heterogeneous bubbles of general geometries and under general boundary conditions. We remark that identifying the
free energy of a continuum medium in an electric field has proved to be quite subtle. Alternative, sometimes contradictory, expressions of the total free energy (or internal energy) of the system (including or not including the effects of boundary devices) have been proposed in the literature. This has generated quite some confusions. The reader is referred to one of the authors’ recent works \cite{13,14} and references therein for the origins and relationship between different energy formulations pertaining to electro-mechanical coupling. Here, for the EHD model of a single bubble on solid substrate, we identify the free energy of the system, carry out the derivation of electric contribution to the mechanical balance of the bubble interface, and obtain the variational principle (a minimization problem, as anticipated) that determines the equilibrium shape of a bubble in an electric field. This variational principle also implies a robust numerical algorithm for computing the equilibrium shape of a bubble. Following the framework of the level-set method \cite{40–42} and sensitivity analysis \cite{43}, we implement a gradient scheme for minimizing the free energy of the system with a given volume and obtain the final equilibrium shape of the bubble. This numerical scheme is further verified by comparing with analytical results \cite{39} at the absence of electric field and experimental results at the presence of electric field.

As is well known, a key difficulty of simulating the equilibrium shape of a bubble in an electric field lies in computing the non-local Maxwell stress on the interface. A second difficulty in a Lagrangian scheme arises from tracking the interface. The level-set based gradient method is particularly convenient to address these issues. Roughly speaking, the level-set function is defined on a fixed mesh to characterize the shape of the bubble; the same mesh is used in the finite-element model for solving the electrostatic problem. We can then determine the driving forces on the interfaces by a sensitivity analysis. By enforcing a kinetic law that relates the normal velocity of the interface and driving force, the evolution of the level-set function follows a Hamilton–Jacobi equation and the total free energy decreases monotonically with respect to (fictitious) time until the final equilibrium shape is reached. We remark that our numerical scheme is particularly robust and suitable for stability analysis of the bubble since it is based on a minimization problem. It is also worthwhile to mention the work of Di Marco et al. \cite{11} where the dynamic process of a growing bubble in an electric field has been studied via a scheme that combines the volume of fluid method, level-set method and ghost fluid approach.

The paper is organized as follows. In §2, we begin with the field equation formulation of the problem including the Maxwell equation for the electric field, the equilibrium equations for the interfaces and the associated boundary conditions. A variational formulation of the problem is presented in §3. We also show the associated Euler–Lagrange equations are equivalent to the field equations in §2 and derive the driving forces on the interfaces that will be useful for our numerical scheme. In §4, following \cite{44,45} we describe the level-set gradient method and the detailed algorithm for minimizing our energy functional. Some numerical examples are given in §5 including comparison with prior experimental data and analytical solutions. We conclude and summarize in §6.

**Notation.** We employ direct notation for brevity if possible. Vectors are denoted by bold symbols, such as \( \mathbf{e}, \mathbf{u} \), etc. When index notations are in use, the convention of summation over repeated index is followed. The inner (or dot) product of two vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \) is defined as \( \mathbf{a} \cdot \mathbf{b} := (a_i)(b_i) \) and \( \mathbf{a} \cdot \mathbf{Mb} = a_iM_{ij}b_j \) for a matrix \( \mathbf{M} \in \mathbb{R}^{3 \times 3} \). From the viewpoint of matrices, the \( i \)th row vector of the gradient of a vector field, e.g. \( \nabla \mathbf{u} \), is the gradient of the \( i \)th component of \( \mathbf{u} \), whereas the ‘div’ operates on the row vectors of a matrix field. Therefore, \( \text{div} \nabla \mathbf{u} = \Delta \mathbf{u} \) and \( \text{div}[(\nabla \mathbf{u})^T] = \nabla(\text{div} \mathbf{u}) \). For a scaling parameter \( \varepsilon \ll 1 \), \( O(\varepsilon) \) implies the asymptotic behaviour \( O(\varepsilon)/\varepsilon \to C \neq 0 \) as \( \varepsilon \to 0 \), whereas \( o(\varepsilon)/\varepsilon \to 0 \) as \( \varepsilon \to 0 \).

### 2. Problem statement

Consider a two-phase immiscible fluid as shown in figure 1 and denote by \( D \subset \mathbb{R}^3 \) the domain occupied by the two-phase fluid. Let \( \Omega \) and \( \Omega^c = D \setminus \Omega \) denote the domain occupied by the
first-phase fluid and the second-phase fluid with the density and permittivity given by

\[ \rho(x) = \begin{cases} 
\rho_1 & x \in \Omega, \\
\rho_2 & x \in D \setminus \Omega 
\end{cases}, \quad \epsilon(x) = \begin{cases} 
\epsilon_1 & x \in \Omega, \\
\epsilon_2 & x \in D \setminus \Omega, 
\end{cases} \]

respectively. We further assume that the domain \( \Omega \) is regular, axisymmetric and open bounded. The fluid phases are in contact with the solid container and, in particular, the first-fluid phase, the second-fluid phase and the solid substrate meet at a line which we denote by \( L_{12s} \). Denote by \( \Gamma_{12} \) the interface between the first- and the second-fluid phase, and by \( \Gamma_{1s} \) (\( \Gamma_{2s} \)) the interface between the first- (second) fluid phase and the solid substrate. Let \( \gamma_{12}, \gamma_{1s} \) and \( \gamma_{2s} \) be the surface tensions of the interfaces \( \Gamma_{12}, \Gamma_{1s} \) and \( \Gamma_{2s} \), respectively. The line tension in the junction line \( L_{12s} \) is denoted by \( l_{12s} \).

We apply an external electric field on the two-phase fluid by maintaining a constant potential difference between two planar electrodes on the boundary of the domain \( D \) (figure 1). The bottom (respectively, top) electrode occupying \( \Gamma_{D1} \subset \partial D \) (respectively, \( \Gamma_{D2} \subset \partial D \)) has the electrostatic potential \( \varphi = V_1 \) (respectively, \( \varphi = V_2 \)). By the Maxwell equation, the electrostatic potential \( \varphi : D \to \mathbb{R} \) necessarily satisfies

\[
\begin{cases}
\text{div}[\epsilon(x)\nabla \varphi] = 0 & \text{in } D, \\
\varphi = V_1 & \text{on } \Gamma_{D1}, \\
\varphi = V_2 & \text{on } \Gamma_{D2} \\
\hat{n} \cdot \nabla \varphi = 0 & \text{on } \partial D \setminus (\Gamma_{D1} \cup \Gamma_{D2}).
\end{cases}
\]

(2.1)

Here, the last boundary condition follows from the assumption that the boundary \( \partial D \setminus (\Gamma_{D1} \cup \Gamma_{D2}) \) is far away from the bubble and hence the fringe field is negligible. We remark that the above boundary-value problem uniquely determines the local electric field.

We seek to determine the equilibrium shape of the bubble \( \Omega \) assuming that the inertial and hydrodynamic forces are negligible. To this end, we first note that the boundary of \( \Omega \) may in general consist of the interface between the two fluids \( \Gamma_{12} \), the interface between the first-phase fluid and the solid substrate \( \Gamma_{1s} \) and the junction line \( L_{12s} \). The equilibrium of any sub-interface \( \Sigma \subset \Gamma_{12} \) implies the classic Young–Laplace equation:

\[
\int_{\partial \Sigma} \gamma_{12} \hat{t} + \int_{\Sigma} (p_1 - p_2 + f_e) \hat{n} = 0 \Rightarrow 2\gamma_{12} H = p_1 - p_2 + f_e \quad \text{on } \Gamma_{12},
\]

(2.2)

where \( \hat{n} \) is the unit outward normal to the interface \( \Gamma_{12} \), \( \hat{t} \) is the unit tangent on \( \Gamma_{12} \) that is perpendicular to and point away from \( \partial \Sigma \), and \( H \) is the mean curvature of the interface \( \Gamma_{12} \).
To derive the second of (2.2), we have noted that by the Stokes theorem (e.g. [46]), for any cotangent vector field \( v \) on \( \Gamma_{12} \),
\[
\int_{\partial \Sigma} v \cdot \hat{t} = \int_{\Sigma} \nabla_{s} \cdot v,
\]
where \( \nabla_{s} \) denotes the surface gradient. Then for any constant unit vector \( \hat{k} \in \mathbb{R}^{3} \), the above identity with \( v = \hat{k} - \hat{n} (\hat{k} \cdot \hat{n}) \) implies that
\[
\hat{k} \cdot \int_{\partial \Sigma} \hat{t} = \int_{\Sigma} \nabla_{s} \cdot [\hat{k} - \hat{n}(\hat{k} \cdot \hat{n})] = -\hat{k} \cdot \int_{\Sigma} \hat{n} \nabla_{s} \cdot \hat{n}.
\]

Inserting the above equation into the first of (2.2), from the arbitrariness of the sub-interface \( \Sigma \), by the definition \( 2H = \nabla_{s} \cdot \hat{n} \), we obtain the differential form of the equilibrium equation, i.e. the second of (2.2).

Further, we assume the two-phase fluid is in a uniform downward gravitational field of strength \( g \). Therefore, within a non-consequential additive constant, e.g. the actual pressure at a point on the plane \( \{y = 0\} \) in the second phase, the pressure in the second phase is given by
\[
p_{2} = -\rho_{2} gy \quad \text{in} \quad D \setminus \Omega, \quad (2.3)
\]
whereas the pressure in the first phase is given by
\[
p_{1} = -\rho_{1} gy + p_{*} \quad \text{in} \quad \Omega. \quad (2.4)
\]

Here the unknown constant \( p_{*} \in \mathbb{R} \) is the pressure difference inside and outside the bubble \( \Omega \) at \( y = 0 \) and depends on the volume of the bubble. Finally, the term \( f_{e} \) is the normal stress on \( \Gamma_{12} \) due to the electric interactions between the fluids and the applied electric field and is given by
\[
f_{e} = \hat{n} \cdot \llbracket T_{MW} \rrbracket \hat{n}, \quad T_{MW} = E \otimes D - \frac{\varepsilon(x)}{2} |E|^{2} I, \quad (2.5)
\]
where \( E = -\nabla \varphi \) is the local electric field. Here and subsequently we denote by \( \llbracket * \rrbracket = (+)|_{+} - (-)|_{-} \) the jump across the interface with + (−) side being the exterior (interior) side of \( \partial \Omega \).

Moreover, the equilibrium of any subinterval on the junction line \( L_{12s} \) implies that
\[
\kappa_{l_{12s}} = \gamma_{2s} + \gamma_{12} \cos \theta_{C} - \gamma_{1s} \quad \text{on} \quad L_{12s}, \quad (2.6)
\]
where \( \theta_{C} \) is the contact angle between the first fluid phase and the solid phase and \( \kappa \) is the curvature of the junction line \( L_{12s} \). The above equation may be regarded as the boundary condition of the Young–Laplace equation (2.2). There are also situations, e.g. the first fluid is ejected into the second fluid from a pipette of radius \( r_{0} \). In this case, it is the junction line that is prescribed and we shall require \( L_{12s} \) coincides with the wall of the pipette:
\[
r(y)|_{y=0} = r_{0}. \quad (2.7)
\]

For a given volume of the bubble \( \Omega \), we anticipate that equations (2.1), (2.2) and (2.6) or (2.7) admit a solution that determines the equilibrium shape of \( \Omega \) for some constant \( p_{*} \). In the absence of electric field and neglecting the line tension, the solutions to (2.1), (2.2) and (2.6) can be obtained by solving an ordinary differential equation, see [30,39]. In the presence of electric field, the non-local Maxwell stress (2.5) makes the analytical and numerical solutions to (2.1), (2.2) and (2.6) or (2.7) challenging problems. Below we present an alternative formulation of the problem that justifies formula (2.5) of Maxwell stress and furnishes a convenient numerical scheme for computing the equilibrium shapes.

3. An equivalent variational formulation

In this section, we reformulate the problem of determining the equilibrium shape of the bubble as an energy minimization problem. This approach clarifies the origin of the formula (2.5) of Maxwell stress on the interface and gives rise to a convenient gradient method for solving (2.1), (2.2) and (2.6) or (2.7).
For the system of two immiscible fluids as shown in figure 1, we describe the thermodynamic state of the system by the shape and size of the bubble $\Omega$ and the polarization $P : D \to \mathbb{R}^3$ in both fluids. In terms of state variables $(\Omega, P)$, we identify the total free energy of the system as

$$\mathcal{E}_{\text{tot}}[\Omega, P] = \int_{\Omega} (\rho_1 - \rho_2)g y + \int_{\Gamma_{12}} \gamma_{12} + \int_{\Gamma_4} (\gamma_{1s} - \gamma_{2s}) + \int_{L_{12s}} l_{12s} + \mathcal{E}^{\text{elect}}[\Omega, P],$$

(3.1)

where the first three terms arise from the gravity, surface tension of the interface between two fluids, and surface tensions between two fluids and solid substrate, respectively. Also, following Liu [13,14], we write the electric part of free energy of the system $\mathcal{E}^{\text{elect}}[\Omega, P]$ as

$$\mathcal{E}^{\text{elect}}[\Omega, P] = \frac{\varepsilon_0}{2} \int_D |\nabla \psi|^2 + \int_{\partial D} \psi(-\varepsilon_0 \nabla \psi + P) \cdot \hat{n} + \int_D \frac{|P|^2}{2(\varepsilon(x) - \varepsilon_0)},$$

(3.2)

where the potential $\psi$ is determined by the boundary value problem (2.1). In the above equation, the first term denotes the energy of the electric field, the second term is contributed by the electric loading device, say, a battery used to maintain the potential difference and the last term arises from polarizing the molecules in the fluids. For a fixed volume $\Lambda_0$ of $\Omega$, by the principle of minimum free energy the equilibrium shape of the bubble shall be dictated by the minimization problem

$$\min\{\mathcal{E}_{\text{tot}}[\Omega, P] : \text{vol}(\Omega) = \Lambda_0, \ P \in \mathbb{P}\},$$

(3.3)

where $\text{vol}(\Omega)$ denotes the volume of the domain $\Omega$ and $\mathbb{P} := \{P : \int_D |P|^2 < +\infty\}$ consists of all square integrable polarizations.

Since the mechanical part of free energy (i.e. the first four terms of (3.1)) is independent of polarization $P$, it will be convenient to a priori solve the minimization problem with respect to polarization $P$ and introduce an effective electric energy that depends only on $\Omega$:

$$\mathcal{E}^e[\Omega] := \min_{P \in \mathbb{P}, \text{vol}(\Omega) = \Lambda_0} \mathcal{E}^{\text{elect}}[\Omega, P].$$

(3.4)

The Euler–Lagrange equations associated with the above minimization problem can be calculated by considering variations of polarization $P \to P + \delta P$. Standard first-variation calculation yields that the minimizing polarization necessarily satisfy

$$P = -(\varepsilon(x) - \varepsilon_0)\nabla \psi \quad \text{in } D,$$

(3.5)

as one would expect from the definition of permittivity and polarization: $D = \varepsilon(x)E = \varepsilon_0 E + P$. Inserting the above equation into (3.2), by the divergence theorem we find the minimum can be written as

$$\mathcal{E}^e[\Omega] = \int_D \frac{\varepsilon_0}{2} |\nabla \psi|^2 + \int_D \frac{\varepsilon(x) - \varepsilon_0}{2} |\nabla \psi|^2 + \int_{\partial D} \psi[-\varepsilon(x) \nabla \psi] \cdot \hat{n} - \frac{1}{2} \int_D \varepsilon(x)|\nabla \psi|^2.$$

(3.6)

By (3.1), (3.4) and (3.6), we rewrite the total free energy of the system as a functional of the bubble shape $\Omega$:

$$\hat{\mathcal{E}}_{\text{tot}}[\Omega] = \int_{\Omega} (\rho_1 - \rho_2)g y + \int_{\Gamma_{12}} \gamma_{12} + \int_{\Gamma_4} (\gamma_{1s} - \gamma_{2s}) + \int_{L_{12s}} l_{12s} - \frac{1}{2} \int_D \varepsilon(x)|\nabla \psi|^2.$$

(3.7)

We claim that equilibrium equation (2.2) for the interface $\Gamma_{12}$ and (2.6) for the junction line $L_{12s}$ can be regarded as the Euler–Lagrange equation of the variational principle (3.3). To see this, without change of notation we denote by $\hat{\Omega}$ a local minimizer of the energy functional $\hat{\mathcal{E}}_{\text{tot}}[\Omega]$ in (3.7). We now consider small variations of the domain $\Omega$ such that the boundary points of the
new domain $\Omega_\delta$ consisting of $y = x + \delta y_1(x)$, where $x \in \partial \Omega$, $y_1 : \partial \Omega \to \mathbb{R}^3$ can be interpreted as the velocity and $\delta$ as a small (fictitious) time interval. Since $\Omega$ is a local minimizer, we infer that for any volume-conserving velocity $y_1$ and any number $\delta \in \mathbb{R}$ small enough,

$$\hat{E}_{\text{tot}}[\Omega_\delta] - \hat{E}_{\text{tot}}[\Omega] =: T_1 + T_2 + T_3 + T_4 + T_5 \geq 0. \quad (3.8)$$

In the above equation, for brevity we have defined the quantities $T_i (i = 1, \ldots, 5)$ as:

$$T_1 = \int_{\Omega_\delta} (\rho_1 - \rho_2)gy - \int_{\Omega} (\rho_1 - \rho_2)gy, \quad T_2 = \int_{\Gamma_{12}^\delta} \gamma_{12} - \int_{\Gamma_{12}} \gamma_{12},$$

$$T_3 = \int_{\Gamma_{12}^\delta} (\gamma_{1s} - \gamma_{2s}) - \int_{\Gamma_{1s}} (\gamma_{1s} - \gamma_{2s}), \quad T_4 = \int_{L_{12s}^\delta} l_{12s} - \int_{L_{12s}} l_{12s},$$

and

$$T_5 = -\frac{1}{2} \int_{\mathcal{D}} \epsilon_\delta(x)|\nabla \varphi_{12}|^2 + \frac{1}{2} \int_{\mathcal{D}} \epsilon(x)|\nabla \varphi|^2,$$

where $\epsilon_\delta(x)$ takes the value of $\epsilon_1$ if $x \in \Omega_\delta$ and $\epsilon_2$ if $x \in \mathcal{D} \setminus \Omega_\delta$, $\varphi_\delta$ is the solution to (2.1) for this new dielectric function $\epsilon_\delta(x)$, $\Gamma_{12}^\delta$ is the new interface between the first and second fluids, $\Gamma_{1s}^\delta$ is the new interface between the first fluid and the solid phase, and $L_{12s}^\delta$ is the new junction line between the first, second fluids and the solid substrate.

The implications of (3.8) depend on the choices of the velocity $y_1$. First of all, to conserve the volume of $\Omega_\delta$ the velocity $y_1$ necessarily satisfies

$$\int_{\partial \Omega} y_1 \cdot \mathbf{n} = 0. \quad (3.9)$$

If the velocity $y_1$ is supported on $\Gamma_{12}$ (which implies the interfaces between the fluids and the solid substrate remain unchanged if $\delta$ is small enough), without loss of generality we can assume the velocity $y_1$ is normal to the interface $\Gamma_{12}$ and given by $y_1 = \nu \mathbf{n}$. Then it is clear that $T_3 = T_4 = 0$,

$$T_1 = \delta \int_{\Gamma_{12}} (\rho_1 - \rho_2)gy + o(\delta), \quad (3.10)$$

and (e.g. [47])

$$T_2 = \delta \gamma_{12} \int_{\Gamma_{12}} 2Hv + o(\delta). \quad (3.11)$$

Moreover, we find that the change of the electrostatic energy $E_{\text{elec}}$ with respect to a small variation of the domain $\Omega$ is given by (see appendix A for details)

$$T_5 = -\delta \int_{\Gamma_{12}} f_e v + O(\delta^2) = -\delta \int_{\Gamma_{12}} v f_e + o(\delta). \quad (3.12)$$

where the stress $f_e$, defined in (2.5), is contributed by electrostatic interaction, i.e. the Maxwell stress. From (3.12), the Maxwell stress on the interface between two fluids can be regarded as a configurational force that determines the change of electric energy of the system when the interface is perturbed.

We remark that the derivation of (3.12) is quite technical, involving variations of phase boundary $\Gamma_{12}$ or ‘inner variation’ of the electrostatic potential $\varphi$, and henceforth postponed to appendix A. For a history and applications of this technique, the interested reader is referred to Eshelby [48] in the context of elasticity for his celebrated energy-moment tensor, Simon [49] for a general concept of shape derivatives, and to recent works of Grabovsky et al. [50], Liu [14] and references therein.

From (3.10)–(3.12) and taking into account the constraint (3.9) by a Lagrangian multiplier $-p_* \int_{\Gamma_{12}} v$, we obtain the Euler–Lagrange equation associated with the variational principle (3.3): \[2 \gamma_{12} H + (\rho_1 - \rho_2)g y - f_e - p_* = 0 \quad \text{on} \quad \Gamma_{12},\] which can be identified with the Young–Laplace equation (2.2).
We now calculate the boundary conditions on the junction line \( L_{12s} \). To this end, we shall consider variations with non-zero velocity on \( L_{12s} \). Without loss of generality, assume that
\[
y_1 = v_x(u_1, u_2)e_x \quad \text{on } \Gamma_{12},
\]
where \( v_x \) decays to zero away from the junction line, \( e_x \) is unit vector in \( x \) direction. Let \( \hat{t} \) be the outward tangential unit vector on the boundary line of \( \Gamma_{12} \). Then
\[
T_1 = \delta \int_{\Gamma_{12}} (\rho_1 - \rho_2)g y v_x e_x \cdot \hat{n} + o(\delta), \quad T_2 = \delta \int_{\Gamma_{12}} 2Hy v_x e_x \cdot \hat{n} + o(\delta)
\]
and
\[
T_3 = \delta \int_{L_{12s}} (\gamma_1 s - \gamma_2 s) v_x + o(\delta), \quad T_4 = \delta \int_{L_{12s}} l_{12s} \kappa v_x + o(\delta).
\]
In addition, by (3.12) we have
\[
T_5 = -\delta \int_{\Gamma_{12}} f v_x e_x \cdot \hat{n} + o(\delta).
\]

For boundary conditions, we shall assume the trial function \( v_x \) decays so quickly away from the boundary that all area integrals in (3.13)–(3.15) are negligible as compared with the line integrals. Therefore, by (3.8) we obtain
\[
l_{12s} \kappa + \gamma_1 s - \gamma_2 s - \gamma_{12} \cos \theta_C = 0 \quad \text{on } L_{12s},
\]
which can be identified with the equilibrium equation (2.6) for the junction line.

Based on the variational formulation (3.3), we now describe the gradient method for computing the equilibrium shape \( \Omega \) with a prescribed volume \( \text{vol}(\Omega) = \Lambda_0 \) satisfying equations (2.2) and (2.6) or (2.7). For an initial guess of the shape \( \Omega_0 \), we let the domain evolve in the descent direction of the total energy and obtain the shape \( \Omega_t \) as a function of the fictitious time \( t \). As shown above, if \( \Omega_t \) is not in equilibrium, then for a small variation
\[
\partial \Omega_t \ni x \to y = x + \delta v \hat{n} \in \partial \Omega_{t+\delta},
\]
the change of total energy is given by
\[
\hat{\mathcal{E}}_{\text{tot}}[\Omega_{t+\delta}] - \hat{\mathcal{E}}_{\text{tot}}[\Omega_t] = -\delta \int_{\Gamma_{12}} v f^{d}_{12} - \delta \int_{L_{12s}} v \hat{n} \cdot e_x f^{d}_{12s} + o(\delta),
\]
where
\[
f^{d}_{12} = -2\gamma_{12} H - (\rho_1 - \rho_2)g y + f_c \quad \text{and} \quad f^{d}_{12s} = -l_{12s} \kappa - \gamma_1 s + \gamma_2 s + \gamma_{12} \cos \theta_C.
\]
may be called the driving force on the boundary \( \partial \Omega_t \) and the junction line \( L_{12s} \), respectively. If we enforce a kinetic law
\[
v = c f^{d}_{12} \quad \text{on } \Gamma_{12} \quad \text{and} \quad v = c f^{d}_{12s} \hat{n} \cdot e_x \quad \text{on } L_{12s},
\]
for some positive constant \( c > 0 \) and let the boundary evolves according to (3.16), by (3.17) the change of total energy is given by
\[
\hat{\mathcal{E}}_{\text{tot}}[\Omega_{t+\delta}] - \hat{\mathcal{E}}_{\text{tot}}[\Omega_t] = -\delta \int_{\Gamma_{12}} c |f^{d}_{12}|^2 - \delta \int_{L_{12s}} c |f^{d}_{12s}|^2 |\hat{n} \cdot e_x|^2 + o(\delta),
\]
and is negative for small enough time step \( \delta \); the domain \( \Omega_t \) is anticipated to converge to an equilibrium shape as \( t \) increases.
4. Numerical algorithm based on the level-set method

The above variational formulation implies a natural fixed mesh scheme to compute the equilibrium shape of the bubble. Following Osher & Sethian [40], we employ the level-set method and use a scalar function \( \psi : D \times [0, +\infty) \rightarrow \mathbb{R} \) to characterize the shape of the bubble \( \Omega_t \):

\[
\psi(x, t) < 0 \quad \Leftrightarrow \quad x \in \Omega_t,
\]

\[
\psi(x, t) = 0 \quad \Leftrightarrow \quad x \in \partial \Omega_t
\]

and

\[
\psi(x, t) > 0 \quad \Leftrightarrow \quad x \in D \setminus \Omega_t.
\]

In account of the axis-symmetry of \( \Omega_t \), we denote by \( (r, y) \) a point in the \( xy \)-plane \( \{(x, y, z) : z = 0 \} \), where \( r = \sqrt{x^2 + z^2} \). Let \( \hat{n} = \nabla \psi / |\nabla \psi| \) be the outward normal on the boundary of the domain \( \{(r, y) : \psi(r, y) \leq \text{const.}\} \). Upon initiating the level-set function \( \psi(r, y, t) \) by the kinetic law (3.18) and let the level-set function accordingly evolve. Since a point \( (r(t), y(t)) \) on \( \partial \Omega_t \) satisfies \( \psi(r(t), y(t), t) = 0 \), differentiating with respect to \( t \) we obtain

\[
\frac{\partial}{\partial t} \psi + v(r, y, t) |\nabla \psi| = 0,
\]

where the velocity field \( v(r, y, t) \) is given by (3.18) when restricted to \( \partial \Omega_t \).

Below we compute the velocity field contributed by various driving forces. In terms of the level-set function \( \psi(r, y, t) \), the mean curvature is given by (cf. [45, eqn (5.33), p. 58])

\[
2H = \nabla \cdot \hat{n} = \frac{(\psi_{yy} + \psi_{rr}/r)\psi_{r}^2 + \psi_y^2(\psi_{rr} + \psi_{rr}/r) - 2\psi_r\psi_y\psi_{ry}}{(\psi_r^2 + \psi_y^2)^{3/2}} = \frac{(x\psi_r/r, \psi_y, z\psi_r/r)}{(\psi_r^2 + \psi_y^2)^{1/2}}, \tag{4.2}
\]

where \( \psi_r, \psi_y \) denote the derivatives of \( \psi \) with respect to \( r, y \), respectively. By the kinetic law (3.18), it contributes a velocity field

\[
v_H(r, y) = -2c_1\gamma_2H(r, y) \quad \forall \ (r, y) \in D. \tag{4.3}
\]

A second velocity field is contributed by the density difference between the two fluids. By (3.10), we have

\[
v_\gamma(r, y) = -c(\rho_1 - \rho_2)gy \quad \forall \ (r, y) \in D. \tag{4.4}
\]

Thirdly, to compute the velocity field contributed by the Maxwell stress

\[
v_e(r, y) = c_f \quad \forall \ (r, y) \in D, \tag{4.5}
\]

we need to a priori solve the Maxwell equation (2.1) and obtain the electric field \( E(r, y) = (\psi_r, \psi_y) \). Then the Maxwell stress defined by (2.5) on the interface \( \Gamma_{12} \) is given by

\[
f_e = \left[ (\hat{n} \cdot E)(\hat{n} \cdot D) - \frac{\epsilon(x)}{2} |E|^2 \right].
\]

Since the interface \( \Gamma_{12} \) is implicit in our scheme, the above formula is replaced by

\[
f_e = s \hat{n} \cdot \nabla f_{MW}, \quad f_{MW} := (\hat{n} \cdot E)(\hat{n} \cdot D) - \frac{\epsilon(x)}{2} |E|^2
\]

and applied to the entire domain \( D \), where \( s \) is the grid size for computing the gradient in the numerical scheme (cf. step 1 in the subsequent description of the algorithm). Finally, to constrain the volume, the velocity field \( v_p \), contributed by the Lagrangian multiplier \( p_s \) shall be such that

\[
\int_{\partial \Omega} (v_p + v_H + v_\gamma + v_e) = 0.
\]

However, in our scheme it is inconvenient to compute integrals over the boundary \( \partial \Omega_t \). We shall simply define

\[
v_{p^*}(r, y) = \begin{cases} 
\delta_1 & \text{if } A_1 < A_0 \\
-\delta_1 & \text{if } A_1 > A_0 
\end{cases} \quad \forall \ (r, y) \in D, \tag{4.6}
\]
where $\Lambda_0$ be the prescribed volume of the first fluid, $\Lambda_t$ is the volume of the domain $\Omega_t := \{(x, y, z) : \psi(x, y) \leq 0\}$ and $\delta_1 > 0$ is a small positive number. Note that the above velocity field $v_p^*$ will induce small fluctuations of the volume of $\Omega_t$. Depending on the stage of simulations, we can adaptively choose the number $\delta_1$ such that the fluctuation is within, e.g. 0.1% of $\Lambda_0$. The overall velocity field is given by
\[ v(r, y) = v_H(r, y) + v_g(r, y) + \psi(r, y, t) + v_p^*(r, y) \quad \forall (r, y) \in D. \quad (4.7) \]

Furthermore, by the second of (3.18) the boundary condition (2.6) can be implemented by introducing
\[ v(r, y)|_{y=0} = c f^d_{12} \hat{n} \cdot e_x, \quad (4.8) \]
whereas the boundary condition (2.6) is enforced by
\[ v(r, y)|_{y=0} = 0. \quad (4.9) \]

We remark that a key advantage of the fixed-mesh level-set method lies in that the interface $\partial \Omega_t$ is implicit in the scheme. In other words, we simulate the evolution of the level-set function $\psi$ instead of the interface itself. Therefore, the velocity field (4.3)–(4.9) are extended to all grid points in the entire computation domain $D$ instead of being restricted to the boundary $\partial \Omega$. The evolution of the level-set function $\psi$ apparently depends on the extension, but the evolution of the domain $\Omega_t$ and the converged equilibrium shape is independent of the particular extension.

With the overall velocity field defined, we summarize our algorithm as follows:

1. Generating a uniform fixed mesh on the spatial domain $D$ of interval size $s_r$ in the $r$-direction and $s_y$ in the $y$-direction. In all subsequent simulations and shown in figure 2, the domain $D$ is a rectangle of ratio $1 : 2$ with a regular mesh of size $100 \times 200$, and hence $s_r = s_y = s$.
2. Initializing the level-set function $\psi^0 = \psi(r, y, t = 0)$. In practice, we always start from a very small hemisphere and let it evolve according to the kinetic law (4.7) to eliminate the potential dependence of the final equilibrium shape on the initialization.
3. For $n \geq 0$, let $\psi^n_j = \psi(r_i, y_i, t_n)$, and $r_i = i s_r$, $y_i = j s_y$, $t_n = n t_1$. Here $t_1 \ll 1$ is the fictitious time step.

Figure 2. The contour plot of the level set function in a typical simulation. The domain $D$ is a rectangle of ratio $1 : 2$ with meshsize $100 \times 200$. The solid red line shows the zero contour representing the interface of the bubble. (Online version in colour.)
(a) Finding the solution $\varphi^n$ to the Maxwell equation (2.1) based on the finite-element method. Computing the overall velocity field $v_{ij}^n$ defined by (4.7)–(4.9). Here, we do not remesh to track the interface; the material properties of each element are either of the first fluid or the second as determined by the level-set function.

(b) Computing the new level-set function $\psi^{n+1}_{ij} = \psi(r_i, y_j, t_{n+1})$ by the Hamilton–Jacobi equation (4.1). In particular, we will use an explicit first-order upwind scheme described in [45, p. 55]:

$$
\frac{\psi^{n+1}_{ij} - \psi^n_{ij}}{t_1} + \max(v_{ij}, 0)g^+_ij + \min(v_{ij}, 0)g^-_{ij} = 0,
$$

where

$$
g^+_ij = \left[ \max(D^{ij}_{y}, 0)^2 + \min(D^{ij}_{y}, 0)^2 \right]^{1/2},
$$

$$
g^-_{ij} = \left[ \max(D^{ij}_{y}, 0)^2 + \min(D^{ij}_{y}, 0)^2 \right]^{1/2},
$$

$$
D^{ij}_{y} = \frac{\psi^n_{ij}(r_{i+1}, y_j) - \psi^n_{ij}(r_{i}, y_j)}{y_{j+1} - y_{j}}, \quad D^{ij}_{y} = \frac{\psi^n_{ij}(r_{i}, y_{j+1}) - \psi^n_{ij}(r_{i}, y_{j})}{y_{j+1} - y_{j}}.
$$

(4) For stability, we reinitialize the level-set function such that it is the signed distance function:

$$
\psi^{n+1}(r, y) = \{ \text{sgn}(\psi^{n+1}) \text{dist}((r, y), \partial \Omega_{t_{n+1}}) : (r, y) \in D \}.
$$

This is an important step in practice since the level-set function may become too steep, resulting in bad approximations to the unit normal $\hat{n}$ and the mean curvature $H$.

(5) Iterating the above step 2–step 4 until the total energy converges.

A few remarks are in order here regarding the present numerical scheme based on the variational formulation. First, for the initialization described above our simulations always yield the same final shapes for a prescribed final volume and electric field, though we cannot rigorously prove the uniqueness of the final solution. Second, the Young–Laplace equation (2.2) coupled with the non-local Maxwell stress could be solved by a self-consistent scheme. Roughly speaking, one could describe the interface $\partial \Omega$ by Lagrangian coordinates and iterate the boundary $\partial \Omega$ until the Young–Laplace equation (2.2) is satisfied. This self-consistent scheme is, to the best of our knowledge, not as efficient as compared with present gradient method since (i) every iteration requires remeshing of the entire domain for solutions of the Maxwell equation (2.1), (ii) the gradient or the ‘optimal’ direction of iteration is not known and (iii) there is no physics-based criterion for convergence and for topological changes of the domain in some more general problems, e.g. splitting or merging of the bubble. Last, though not addressed in the present work, the variational formulation can yield stability criterion for the equilibrium shape by computing the second variations of the total free energy [50] and the numerical scheme can capture the onset of departure of the bubble from the substrate.

5. Numerical examples and validation

(a) Comparison with analytical results

We verify our scheme by comparing with analytical results of Cheng & Chaddock [30] in two scenarios. First, at the absence of electric fields, Cheng & Chaddock [30] have shown that the equilibrium shape $y = y(r)$ satisfies the following dimensionless nonlinear ordinary differential
and use the boundary condition (2.7) with \( r \) determined by \( R \) where \( 10 \) the substrate has a radius \( 5.1 \) mm, respectively. The solid line represents the equilibrium shapes implied by (5.1); the dotted line is obtained by (5.3); the diamond points are our simulation results. Also, the inset shows the analytical solutions and simulation results for \( \alpha \) are in excellent agreement which verifies our numerical approach, both at the absence of electric field.

Figure 3a–d shows the analytical solutions and simulation results for \( \Lambda_0 = 17.1, 11.8, 8.1 \) and \( 5.1 \) mm, respectively. The solid line represents the equilibrium shapes implied by (5.1); the dotted line represents the simulated equilibrium shapes. From figure 3, we see that the simulated and analytical equilibrium shapes are in excellent agreement which verifies our numerical approach at the absence of electric field.

Second, in the presence of electric field and neglecting gravity, by assuming the shape would remain as prolate spheroids Cheng & Chaddock [30] have shown that the aspect ratio \( \alpha := \) long-axislength/short-axislength of the spheroid in a uniform external electric field \( \vec{E} = E e_y \) are determined by

\[
\frac{\partial}{\partial \alpha} \left( \alpha^{-2/3} + \alpha^{1/3} \sin^{-1} e \right) - \frac{\epsilon_0 (E e_y^3) \gamma_{12} \partial H}{3 \gamma_{12} \partial \alpha} = 0, \tag{5.3}
\]

where \( \gamma_{12} \) is the surface tension on the external electric field: the solid line is obtained by (5.3); the diamond points are our simulation results. Also, the inset shows the simulated equilibrium shapes of the bubble in different external fields, which shows that the equilibrium shapes indeed look much like spheroids and verifies the assumption of Cheng & Chaddock [30]. Also, the aspect ratios’ dependence on the external electric field from simulations agrees well with the prediction of (5.3).

Figure 4 shows the dependence of the aspect ratio \( \alpha \) on the external electric field: the solid line is obtained by (5.3); the diamond points are our simulation results. Also, the inset shows the simulated equilibrium shapes of the bubble in different external fields, which shows that the equilibrium shapes indeed look much like spheroids and verifies the assumption of Cheng & Chaddock [30]. Also, the aspect ratios’ dependence on the external electric field from simulations agrees well with the prediction of (5.3).

In summary, from figures 3 and 4 we see that the simulated and analytical equilibrium shapes are in excellent agreement which verifies our numerical approach, both at the absence of electric fields and in the presence of electric fields. In the next section, we will compare our simulations with experimental results in the presence of an external electric field and gravity.

(b) Comparison with experiments

In the presence of electric field, analytical solutions to the equilibrium shapes are not available. We validate our numerical approach by the experimental results obtained by Chen et al. [29]. In the experiments, a single nitrogen bubble is injected into transformer oil through an orifice of radius 0.75 mm. A voltage is applied between the top and bottom electrodes. The densities of two phases and surface tension are given by (5.2). In addition, the electric permittivity of two phases are given by

\[
\epsilon_1 = \epsilon_0 \quad \text{and} \quad \epsilon_2 = 2.1 \epsilon_0,
\]

More details about the experimental set-up can be found in Chen et al. [29].
The volume of the nitrogen bubble in the transformer oil increases as nitrogen is being injected. A high-speed camera records the bubble formation and evolution in a DC electric field. From the recorded images of the bubble (cf. the inset photos in figure 5), we can find the contour of the bubble and the volume $\Lambda_0$ of the bubble by MATLAB. We then compute the equilibrium shape of the bubble for the prescribed electric voltage and volume $\Lambda_0$ by the algorithm described in §4. Figure 5 shows the experimental results and simulated equilibrium shapes in an external electric field of 2 kV mm$^{-1}$ (figure 5a,b) and 3 kV mm$^{-1}$ (figure 5c,d). The volumes of the bubbles are fixed by the experimental images and given by 1.56 mm$^3$ (a), 6.20 mm$^3$ (b), 1.80 mm$^3$ (c) and 6.10 mm$^3$ (d). The solid line represents the shape of the simulation results and the dotted line represents the shape of the experimental results. From figure 5a–d, we see that the simulation results agree well with the experiment which validates our numerical approach.

During the simulations, we also monitor the convergence of the total energy as shown in figure 6 for figure 5b. For all simulations, the initial shape of the bubble is chosen as a hemisphere of radius $r_0 = 0.75$ mm. For a final volume $\Lambda_0 > (2\pi/3)r_0^3$, by (4.6) the volume and total energy of the bubble increases at the beginning until reach $\Lambda_0$. Then by (4.6) the volume of the bubble fluctuate negligibly around $\Lambda_0$ while the shape of the bubble is driven to minimize the total free energy.

Figure 3. Comparison of analytical and simulated equilibrium shapes of the bubble at the absence of electric field: the solid line represents the equilibrium shapes implied by (5.1); the dotted line represents the simulated equilibrium shapes. The volume of the bubble is 17.1, 11.8, 8.1 and 5.3 mm$^3$ in (a), (b), (c) and (d), respectively. (Online version in colour.)
Figure 4. Aspect ratio $\alpha$ as a function of dimensionless electric field strength $\varepsilon_0 (E_{\text{eff}})^2 r_{\text{eff}}/\gamma_1$2. The solid line represents the analytical solutions implied by (5.3); the diamond points represent the simulated results. The inset picture is the corresponding simulated equilibrium shape of the bubble under different electric fields 2.0, 4.0, 8.0, 12.0 and 16.0 kV mm$^{-1}$, respectively. (Online version in colour.)

Figure 5. Comparison simulation result with experiment. The solid line represents the shape of the simulation result and the dotted line represents the shape of the experiment result. The volume is 1.56, 6.20, 1.10 and 6.10 mm$^3$ from (a) to (d), respectively. Electric field 2 kV mm$^{-1}$ is applied on (a) and (b) and electric field 3 kV mm$^{-1}$ is applied on (c) and (d). The inset pictures in (a)–(d) are reproduced from Chen et al. [29] with Elsevier permission. (Online version in colour.)
energy as shown in figure 6. For a typical case, the total energy converges after a few hundreds of iterations. We remark that the convergence of the total energy is also observed for all cases in figures 3, 5 and 7.

The numerical scheme developed here can be conveniently used for simulating equilibrium shapes of bubbles of different materials and at different environments. As an example, we consider the injection of nitrogen bubble into transformer oil at zero gravity. This can be achieved by simply setting $g = 0$ in our simulation. Figure 7a–c shows the simulated equilibrium shapes for a fixed contact radius of $r_0 = 0.75$ mm and different volumes at 3.0, 6.0 and 10.0 mm$^3$. The electric fields vary from 0.0, 3.0, 6.0 to 9.0 kV mm$^{-1}$. From figure 7, we observe that the equilibrium shape is elongated as the intensity of electric field increases. Based on these simulation results, we can find the relationship among bubble volume, surface area and the applied electric field. Among many other applications, these results can be further used to develop model of EHD enhancement of boiling at zero gravity and predict rate of heat transfer.

6. Summary and discussion

We have developed a variational formulation for the equilibrium shape of heterogeneous bubbles in an electric field and shown that it is equivalent to the classic field equation approach based on the Young–Laplace equation and the concept of Maxwell stress. Based on this energy formulation, we implement a fixed mesh level-set gradient method for simulating the equilibrium shapes of
the bubble in an electric field. The numerical scheme is validated by comparing with analytical solutions and experimental results in some different scenarios. We anticipate that the variational formulation and numerical scheme will find broad applications in areas of EWOD, EHD and electro-deformation of soft materials among others.

Data accessibility. The original MATLAB code is available at https://www.dropbox.com/sh/7z4z1mcv4lwkpbc3/ AAC2-pWu-ELEd0GgINuX2Bva?dl=0.

Authors’ contributions. L.L. and D.L. conceived the research. H.W. and L.L. created the mathematical formulation and conducted the numerical simulations. H.W., L.L. and D.L. wrote the manuscript and all authors analysed and discussed the results.

Competing interests. We declare we have no competing interests.

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Appendix A. Derivation of Maxwell stress

We now proceed to details of deriving (3.12). First, we extend the deformation \( y(x) \) such that the deformation \( y(x) = x + \delta y_1(x) \) is continuously differential on the entire domain \( D \), equal to \( x \) slightly away from \( \partial \Omega \) (i.e. \( |y_1| \) decays quickly to zero away from \( \partial \Omega \)). Moreover, we note that the new potential \( \psi_\delta \) is defined by the Maxwell equation (2.1) with \( \Omega \) replaced by \( \Omega_\delta \). Let

\[
\begin{align*}
F_\delta &= \nabla y = F + \delta F_1 + o(\delta), & F &= \nabla x = I, & F_1 &= \nabla y_1, \\
J_\delta &= J[1 + \delta \text{Tr}(F_1)] + o(\delta), & C_\delta &= F_\delta^T F_\delta = I + \delta F_1 + o(\delta).
\end{align*}
\]

(A 1)

Upon a change variables \( y \to x \), we find the new electrostatic potential \( \psi_\delta \) is also determined by

\[
\begin{align*}
\text{div}(-\epsilon(x)J_\delta C_\delta \nabla \psi_\delta) &= 0 \quad \text{in } D, \\
\psi_\delta &= V_1 \quad \text{on } \Gamma_{D1}, & \psi_\delta &= V_2 \quad \text{on } \Gamma_{D2} \\
\hat{n} \cdot \nabla \psi_\delta &= 0 \quad \text{on } \partial D \setminus (\Gamma_{D1} \cup \Gamma_{D2}).
\end{align*}
\]

(A 2)

In addition, the electric energy (3.6) can be written as

\[
E_{\text{elect}}^\psi[\Omega_\delta] = -\frac{1}{2} \int_D \epsilon(x) J_\delta [\nabla \psi_\delta]^T \nabla \psi_\delta. \tag{A 3}
\]

To calculate the first variation of the electric energy \( E_{\text{elect}}^\psi[\Omega_\delta] \), we assume the new electric potential can be written as

\[
\psi_\delta = \psi + \delta \varphi_1 + o(\delta). \tag{A 4}
\]

Comparing (A 2) with (2.1), we find that the potential \( \varphi_1 \) shall satisfy

\[
\begin{align*}
\text{div}[-\epsilon(x)\nabla \varphi_1 - \epsilon(x) \text{Tr}(F_1) \nabla \psi + \epsilon(x)(F_1 + F_1^T) \nabla \varphi] &= 0 \quad \text{in } D, \\
\varphi_1 &= 0 \quad \text{on } \Gamma_{D1}, & \varphi_1 &= 0 \quad \text{on } \Gamma_{D2} \\
\hat{n} \cdot \nabla \varphi_1 &= 0 \quad \text{on } \partial D \setminus (\Gamma_{D1} \cup \Gamma_{D2}).
\end{align*}
\]

(A 5)

Further, by (A 3) and (A 1) we find

\[
E_{\text{elect}}^\psi[\Omega_\delta] = E_{\text{elect}}^\psi[\Omega] + \delta \text{Var1} + o(\delta), \tag{A 6}
\]

where

\[
\text{Var1} = -\frac{1}{2} \int_D \epsilon(x) [\text{Tr}(F_1)] |\nabla \psi|^2 - \nabla \psi \cdot (F_1 + F_1^T) \nabla \varphi + 2 \nabla \varphi \cdot \nabla \varphi_1. \tag{A 7}
\]

By (A 5) and the divergence theorem, we obtain

\[
\int_D \epsilon(x) \nabla \varphi \cdot \nabla \varphi_1 = \int_D [\text{div}(\varphi_1 \epsilon(x) \nabla \psi) - \varphi_1 \text{div}(\epsilon(x) \nabla \psi)] = 0.
\]
Therefore, (A 7) can be rewritten as

\[ \text{Var} 1 = \int_D \mathbf{F}_1 \cdot \mathbf{T}_{MW}, \quad \mathbf{T}_{MW} = \nabla \varphi \otimes (\varepsilon(x) \nabla \varphi) - \frac{\varepsilon(x)}{2} |\nabla \varphi|^2 \mathbf{I}, \quad (A 8) \]

where \( \mathbf{T}_{MW} \) is precisely the ‘Maxwell stress’ for a linear medium. From the Maxwell equation (2.1), it is clear that

\[ \text{div} \mathbf{T}_{MW} = 0 \quad \text{in} \, D \setminus \partial \Omega, \]

which means the Maxwell stress has a body force contribution at an interior point in the two fluids. However, for the interface \( \Gamma_{12} \), by the divergence theorem we have

\[ \text{Var} 1 = -\int_{\partial \Omega} y_1 \cdot \mathbf{[T}_{MW]} \hat{n} = -\int_{\partial \Omega} v \hat{n} \cdot \mathbf{[T}_{MW]} \hat{n} = -\int_{\partial \Omega} v_{fe}, \quad (A 9) \]

which completes the proof of (2.5).

References


