Damping Parameter Design in Structural Systems Using an Analytical $H_\infty$ Norm Bound Approach

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Abstract—This paper examines the damping parameter design problem for structural systems with collocated measurements and inputs so that $H_\infty$ norm bound constraints are satisfied. We utilize a particular solution of the Bounded Real Lemma that provides an explicit upper bound of the $H_\infty$ norm of a collocated structural system. Using this upper bound result the damping design is formulated as a linear matrix inequality (LMI) optimization problem with respect to the damping coefficients of the structural system. Numerical examples demonstrate the benefits and computational advantages of the proposed damping design method.

I. INTRODUCTION

As an important component of passive structural systems design, the problem of damping parameters design has been studied over a century. The use of dashpots and tuned mass dampers (TMD) are two widely used methods of passive vibration attenuation. These simple devices have been proven very effective for reducing severe vibrations of machinery, buildings, bridges and many other mechanical systems with relatively low cost [2]. There are numerous optimization-based methods that have been proposed for optimal placement and optimal value assignment of such damping devices, yet many of them are not widely adopted due to their complexity and computational inefficiency [3][4][5][6][7]. The problem becomes an extremely challenging one for large-scale structural systems with hundreds or thousands degrees of freedom where the above methods become prohibitive.

Recently, structural parameter design techniques have been proposed by viewing the structural parameter selection and optimization of passive mechanical components as a control system design problem. This approach allows the use of rigorous system theoretic and system mathematical tools to quantify system performance and to provide a connection to closed-loop response in controlled structural systems. Thus, optimal and robust control techniques, such as the $H_2$/LQR and $H_\infty$ optimization approaches, have been proposed for structural parameter design. The state space $H_\infty$ norm control method based on the standard Riccati equation approach or the linear matrix inequality (LMI) formulation are now well developed control synthesis tools. The optimal state feedback and full-order dynamic output feedback $H_\infty$ control synthesis problems can be solved using iterations on the corresponding Riccati solutions or via the computational solution of a convex LMI optimization problem [13][14][15].

On the other hand, the static output feedback and the fixed-order dynamic output feedback $H_\infty$ control synthesis problems are difficult computational problems since they require the solution of (nonconvex) bilinear matrix inequalities (BMIs) or LMI’s with coupling rank constraints [16][17]. The optimal dynamic output feedback $H_2$ control synthesis problems is connected to the solution of the LQG problem, which is a combination of state estimation and state feedback control. However, the use of such control-oriented methods for structural parameter design still leads to complex numerically cumbersome optimization problems[26].

Having strain or displacement information as the system output and feedback of the measured output rate has been proved to be very effective in adding local damping to the structure. The control of structural systems with collocated sensors and actuators has been shown to provide great advantages from a stability, passivity, robustness and an implementation viewpoint. For example, collocated control can easily be achieved in a space structure when an attitude rate sensor is placed at the same location as a torque actuator [1][8]. Collocation of sensors and actuators leads to symmetric transfer functions. Several other classes of engineering systems, such as circuit systems, chemical reactors and power networks, can be modelled as systems with symmetric transfer functions. Stabilization, robustness, model reduction and control of such systems has been examined recently [9][10][11].

In this work, we use recently developed control-oriented algebraic tools to formulate the damping coefficient design problem as an efficient convex linear matrix inequality (LMI) optimization problem. By exploiting the particular structure of collocated structural systems, explicit upper bounds for the $H_\infty$ norm of the system can be obtained. To this end, particular solutions of the Bounded Real Lemma are used and explicit expression for an upper bound of the $H_\infty$ norm of such external symmetric system are obtained that require only the computation of the maximum eigenvalue of a symmetric matrix. Subsequently, we damping parameter design problem is formulated as an LMI optimization problem of minimizing the $H_\infty$ norm bound with respect to the unknown damping coefficients.

The standard notation $> (<)$ is used to denote the positive (negative) definite ordering of symmetric matrices. The $i$th eigenvalue of a real symmetric matrix $N$ will be denoted by $\lambda_i(N)$ where the ordering of the eigenvalues is defined as $\lambda_{\max}(N) = \lambda_1(N) \geq \lambda_2(N) \geq \ldots \geq \lambda_n(N)$. The maximum
singular value of a (not necessarily square) matrix \( N \) will be denoted by \( \sigma_{\text{max}}(N) \), which is also its spectral norm \( \|N\| \).

II. Analytical \( H_{\infty} \)-Norm Bound for Collocated Systems

Consider the following vector second-order representation of a structural system with collocated velocity measurements and inputs

\[
\begin{align*}
M \ddot{q} + D \dot{q} + Kq &= Fu \\
y &= F^{T} \dot{q}
\end{align*}
\]  

(1)

where \( q(t) \in \mathbb{R}^n \) is the generalized coordinate vector, \( u(t) \in \mathbb{R}^m \) is the input vector and \( y(t) \in \mathbb{R}^k \) is the measured output vector. The matrices \( M, D \) and \( K \) are symmetric positive definite matrices that represent the structural system mass, damping and stiffness distribution, respectively.

The system has a state-space realization as follows

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]  

(2)

with

\[
A = \begin{bmatrix} 0 & I \\
- M^{-1}K & - M^{-1}D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\
F^{T} \end{bmatrix}
\]  

(3)

where \( x = [q \quad \dot{q}]^{T} \). Notice that the transfer function \( G(s) \) of the system (2)-(3)

\[
G(s) = sF^{T} (Ms^{2} + Ds + K)^{-1} F
\]

is symmetric, i.e., \( G(s) = G^{T}(s) \). The system (2)-(3) is an externally symmetric state-space realization, that is, there exists a nonsingular matrix \( T \) such that

\[
A^{T}T = TA, \quad C^{T} = TB
\]  

(4)

This class of systems is more general than the class of internally or state space symmetric systems that they satisfy the symmetry conditions (4) with a positive definite transformation matrix \( T \) [12]. Obviously, state-space symmetry implies external symmetry, but the converse is not true, that is, there exist symmetric transfer matrices for which there is no internally symmetric realization. An analytical solution of the \( H_{\infty} \) control problem for internally symmetric systems has been presented in [18].

Recall that the \( H_{\infty} \) norm of the system (2) is given by

\[
\|G\|_{\infty} = \text{sup } \sigma_{\text{max}}\{G(j\omega)\}
\]  

(5)

where \( G(s) = C(sI-A)^{-1}B \) is the transfer function of the system and \( \sigma_{\text{max}} \) denotes the maximum singular value of a matrix. It is well known that for a stable LTI system, its \( H_{\infty} \) norm can be approximated iteratively, for example using a bisection method [19]. The following result shows that for a vector second-order realization (2)-(3), an upper bound of its \( H_{\infty} \) norm can be computed using a simple explicit formula [20].

**Theorem 1** Consider the vector second-order system realization (2)-(3). The system has an \( H_{\infty} \) norm \( \gamma \) that satisfies

\[
\gamma < \tilde{\gamma} = \lambda_{\text{max}}(F^{T}D^{-1}F)
\]  

(6)

To prove this result recall the Bounded Real Lemma (BRL) characterization of the \( H_{\infty} \) norm of a system.

**Lemma 1** [22] A stable system (2) has an \( H_{\infty} \) norm less than or equal to \( \gamma \) if and only if there exists a matrix \( P \geq 0 \) satisfying

\[
\begin{bmatrix}
A^{T}P + PA & PB & CT \\
B^{T}P & -\gamma I & 0 \\
C & 0 & -\gamma I
\end{bmatrix} \leq 0
\]

(7)

Recall also the following Schur complement formula [23].

**Lemma 2** The block matrix

\[
S = \begin{bmatrix}
S_{11} & S_{12} \\
S_{12}^{T} & S_{22}
\end{bmatrix}
\]

where \( S_{11} \) and \( S_{22} \) are symmetric, is positive definite if and only if

\[S_{11} > 0 \quad \text{and} \quad S_{22} - S_{12}S_{11}^{-1}S_{12} > 0\]

or

\[S_{22} > 0 \quad \text{and} \quad S_{11} - S_{12}S_{22}^{-1}S_{12}^{T} > 0\]

Theorem 1 follows from the BRL condition and the following algebraic result [21].

**Lemma 3** Consider matrices \( \Gamma \) and \( Q \) such that \( \Gamma \) has full column rank and \( Q \) is symmetric positive definite. Then \( Q \geq \Gamma^{T}\Gamma \) if and only if

\[
\lambda_{\text{max}}(\Gamma^{T}Q^{-1}\Gamma) \leq 1
\]

**Proof of Theorem 1** (Sketch) The result follows from the Bounded Real Lemma 1 by utilizing the following Lyapunov matrix

\[
P = \begin{bmatrix}
K & 0 \\
0 & M
\end{bmatrix}
\]

(8)

Application of the Schur complement formula in Lemma 2 results in the following condition

\[-D + \frac{1}{\gamma}FF^{T} \leq 0.\]

(9)

Then, application of the Lemma 3 provides the bound (6).Numerical examples in [20] demonstrate the validity and computational efficiency of the above analytical bound.
III. DAMPING DESIGN USING THE ANALYTICAL BOUND APPROACH

The analytical $H_{\infty}$ norm upper bound (6) of the collocated structural system (1) is solely dependent on the damping distribution matrix $D$ and the input/output distribution matrix $F$. For lumped parameter systems, the damping metric $D$ can be expressed in terms of the elemental damping coefficients as follows

$$D = \sum_{i=1}^{m} c_i T_i$$  \hspace{1cm} (10)

where $c_i$ denotes the viscous damping constant of the $i$th damper and $T_i$ represents the distribution matrix of the corresponding damper in the structural system. The $T_i$'s are given symmetric matrices with elements 0, 1 and $-1$ that define the structural connectivity of the damping elements in the structure. Then, using the Schur complement formula, the $H_{\infty}$ norm upper bound condition (6) can be re-written as

$$\left[ \begin{array}{cc} \sum_{i=1}^{m} c_i T_i & F \\ F^T & \gamma I \end{array} \right] \geq 0$$  \hspace{1cm} (11)

Practical structural system design specifications impose upper bound constraints on the values of the damping coefficients, that is

$$0 \leq c_i \leq c_{\text{max}}$$ \hspace{1cm} (12)

Also, often an upper bound on the total available damping resources is imposed, that is

$$\sum_{i=1}^{m} c_i \leq c_{\text{total}}.$$ \hspace{1cm} (13)

Based on the above discussion the damping design problem can be formulated as follows.

Damping Design Optimization Problem

Consider the collocated structural system (1) with the damping distribution (10). For a given positive scalar $\gamma$, the $H_{\infty}$ norm of the system is less than $\gamma$ if the following conditions with respect to the damping coefficients $c_i$ are feasible.

$$\left[ \begin{array}{cc} \sum_{i=1}^{m} c_i T_i & F \\ F^T & \gamma I \end{array} \right] \geq 0$$ \hspace{1cm} (14)

$$0 \leq c_i \leq c_{\text{max}}$$ \hspace{1cm} (15)

$$\sum_{i=1}^{m} c_i \leq c_{\text{total}}.$$ \hspace{1cm} (16)

The above conditions constitute an LMI feasibility problem with respect to the damping coefficients $c_i$. Then the optimization of damping coefficients can be achieved by solving the LMI optimization problem

$$\min_{c_i} \gamma,$$ \hspace{1cm} (17)

subject to constraints (14)-(16).

Remark 1: Some dynamical system can exhibit negative values of damping. This has been observed in the optimal damping design of cables[25]. In such case, the design parameter $c_i$s are allowed to be negative.

The upper bound approach in the above LMI formulation provides a computationally efficient method to compute the damping coefficients of collocated structural systems. Since the assigned upper bound $\gamma$ is always greater than or equals to the exact $H_{\infty}$ norm of the system, the design result is conservative. However, our computational examples and experience with the proposed bound indicate that it indeed provides a good approximation of the exact $H_{\infty}$ norm.

IV. NUMERICAL EXAMPLES

Single Degree of Freedom (1-DOF) Case

To demonstrate and motivate the above results consider the 1-DOF case ($n = 1$) where $q(t), u(t)$ and $y(t)$ are scalar quantities in (1). For this scalar case the magnitude of the frequency response function (FRF) of the system (1) is

$$|G(j\omega)| = \frac{F^2 |\omega|}{\sqrt{(K - M \omega^2)^2 + D^2 \omega^2}}$$

where $M, D$ and $K$ represent the scalar mass, damping and stiffness coefficients of the system. It can be easily observed that at the natural frequency of this dynamic system, i.e.,

$$\omega = \sqrt{\frac{K}{M}}$$

the magnitude of the FRF reaches its maximum. Thus the above FRF magnitude satisfies the following bound at all frequencies

$$|G(j\omega)| \leq \frac{F^2 |\omega|}{D |\omega|} = \frac{F^2}{D}$$

that is, $||G||_{\infty} \leq F^2/D$. This bound is precisely the one provided in Theorem 1 for the system. In fact, in this scalar case the above bound provides the exact $H_{\infty}$ norm of the system, that is $||G||_{\infty} = F^2/D$. Therefore, the $H_{\infty}$ norm bound $||G||_{\infty} \leq \gamma$ is achieved if and only if the damping coefficient $D$ is selected to satisfy the bound

$$D \geq F^2/\gamma$$

Notice that this result coincides with the bound obtained from (14).

As an example consider the case where $F = 10, M = 30 kg$ and $K = 500 N/m$. Then, for a desired $H_{\infty}$ norm bound $\gamma = 0.5$ the designed value of the damping coefficient $D = 10^2/0.5 = 200 N \cdot s/m$. This result is confirmed from the Bode diagram of the system shown in Fig. 1 that obtains a maximum of $20 \log_{10}(0.5) = -6.02$ dB at $\omega = \sqrt{K/M} = 4.08$ rad/sec.

Two-story Shear Building Model

Now let us consider a two-story shear building model with added viscous dampers as shown in Fig. 2. This model is considered in reference [4]. The primary system of mass $m_1 = m_2 = 100$ has stiffness $k_1 = k_2 = 200$ relative to the base. The system is subject to the disturbance inputs $d_1$ and $d_2$ (such as wind gust disturbance on the structure). Our
The design parameters are $c_1$ and $c_2$ and the damping distribution matrices $T_1$ and $T_2$ are obtained from the expansion

$$D = \left[ \begin{array}{cc} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{array} \right] = \sum_{i=1}^{2} c_i T_i = c_1 \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] + c_2 \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right]$$

In addition, the damping coefficients are subject to the constraints

$$c_i \leq 10, \quad i = 1, 2.$$ 

Using the above parameters we formulate the damping design LMIs and solve the corresponding LMI feasibility problem (14)-(15). The damping coefficients are obtained as $c_1 = 5.04$, $c_2 = 6.49$ and the exact $H_{\infty}$ norm of the damped system is 0.481 which is a bit less than the desired bound 0.5. The frequency responses from the disturbance $d_1$ to output $v_1$, $d_2$ to $v_2$, $d_1$ to $v_2$ and $d_2$ to $v_1$ are shown in Fig. 3 and Fig. 4 respectively. It can be seen that the designed system reduced the effect of disturbances significantly and our design goal is achieved.
Five Spring-mass-damper System Example

Next consider a serial mass, spring and damper system shown in Fig. 5. This example is borrowed and revised from the papers by Sipola et al. [26] and Zuo et al. [3].

Our design objective is to optimize the values of the damping coefficients $c_i$ ($i = 1, ..., 5$) so that the $H_\infty$ norm of the collocated system from the disturbance forces $d_1$ and $d_2$ to the velocities of masses $m_2$ and $m_4$ is minimized. The physical constants of this system are $m_i = 1$, $k_i = 1$ and the damping coefficients $c_i$ ($i = 1, ..., 5$) are chosen as the design parameters.

The damping distribution matrix $D$ of this system is given by

$$D = \begin{bmatrix} c_1 + c_2 & -c_2 & -c_3 & -c_4 & -c_5 \\ -c_2 & c_2 + c_3 & -c_3 & -c_4 & -c_5 \\ -c_3 & -c_3 & c_3 + c_4 & -c_4 & -c_5 \\ -c_4 & c_4 + c_5 & -c_4 & c_5 \\ -c_5 & c_5 & c_5 & c_5 & c_5 \end{bmatrix}$$

$$= \sum_{i=1}^{5} c_i T_i.$$

For comparison and design trade-off purposes we consider a family of optimal damper designs using the result of Theorem 2. The design correspond to different values of the total damping capacity $c_{total}$ ranging from 0.5 to 20. The results of the optimal design using the $H_\infty$ upper bound optimization approach are shown in Fig. 6 and Fig. 7. Fig. 6 shows the values of the $H_\infty$ norm bound obtained using our upper bound optimization approach, as well as, the exact $H_\infty$ norm that corresponds to each design as the total damping capacity $c_{total}$ changes.

Indeed, we observe that in each design the guaranteed $H_\infty$ norm bound of the designed system and its actual $H_\infty$ norm are very close. This result demonstrates that our upper bound optimization approach provides a very good upper bound estimate of the actual $H_\infty$ norm of the system. The values of the optimal damping parameter that correspond to each design are shown in Fig. 7. We observe that for each design $c_1 = c_3$ and $c_2 = c_4$ although $c_5$ is closed to 0.

The significance and benefits of the proposed $H_\infty$ upper bound damping parameter design approach is evident in the design and optimization of very large scale structural systems where standard methods based on nonlinear optimization approaches are computationally prohibitive. The proposed LMI-based convex optimization formulation can address the address the design of structural systems with hundreds or thousands of states and design variables. To solve a similar design problem as stated in Ref. [3], it takes 1070 LMI iterations to find a sub-optimal $H_\infty$ solution using alternative minimization. By using Theorem 2, it takes 14 LMI iterations within one second CPU time to find the optimal result. It can be seen that this method is of time efficient compared with other existing methods (although the method proposed by Zuo et al. [3] can design the damping and stiffness parameters simultaneously).

V. CONCLUSION

We have obtained a simple explicit expression for an $H_\infty$ norm bound of structural systems with collocated sensors and actuators. By utilizing these bounds, LMI conditions are formulated to design the damping parameters that guarantee a desired $H_\infty$ norm of such structural systems. The design methods are particularly useful for very large scale systems where the existing damping coefficient design methods are computationally prohibitive.
REFERENCES


