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Alternating Convex Projection Methods for Discrete-Time Covariance Control Design[1]

K. M. GRIGORIADIS[2] AND R. E. SKELTON[3]

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Abstract. The problem of designing a controller for a linear, discrete-time system is formulated as a problem of designing an appropriate plant-state covariance matrix. Closed-loop stability and multiple-output performance constraints are expressed geometrically as requirements that the covariance matrix lies in the intersection of some specified closed, convex sets in the space of symmetric matrices. We solve a covariance feasibility problem to determine the existence and compute a covariance matrix to satisfy assignability and output-norm performance constraints. In addition, we can treat a covariance optimization problem to construct an assignable covariance matrix which satisfies output performance constraints and is as close as possible to a given desired covariance. We can also treat inconsistent constraints, where we look for an assignable covariance which best approximates desired but unachievable output performance objectives; we call this the infeasible covariance optimization problem. All these problems are of a convex nature, and alternating convex projection methods are proposed to solve them, exploiting the geometric formulation of the problem. To this end, analytical expressions for the projections onto the covariance assignability and the output covariance inequality constraint sets are derived. Finally, the problem of designing low-order dynamic controllers using alternating projections is discussed, and a numerical technique using alternating projections is suggested for a solution, although convergence of the algorithm is not guaranteed in this case. A control design example for a fighter aircraft model illustrates the method.

Key Words. Linear systems, control system design, numerical methods, alternating projections.

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[2]Assistant Professor, Department of Mechanical Engineering, University of Houston, Houston, Texas.

[3]Professor, Space Systems Control Laboratory, Purdue University, West Lafayette, Indiana.

1. Introduction

Customarily, the control design problem in state space is posed in terms of the immediate unknowns of the problem, namely the desired controller parameters. Although this is the most direct approach, it has several drawbacks. The control design problem subject to desired performance constraints often results in a nonlinear and nonconvex optimization program with no guaranteed convergence nor global optimality. Moreover, an initial feasible stabilizing controller is usually needed to initiate the algorithm, which sometimes is difficult to obtain depending on the desired design constraints.

In the frequency domain, the Youla–Kucera parametrization of all stabilizing controllers for discrete-time systems (Refs. 1, 2) provides the possibility to transform the control design problem to a problem of finding a proper stable transfer matrix $Q(z)$ such that the control design objectives are met. This approach has been adopted, for example, in Boyd et al. (Refs. 3, 4), where it was shown that the control design problem subject to certain performance objectives can be expressed as a convex optimization problem in terms of the parameter $Q(z)$. Since the underlying space is infinite-dimensional, only an approximating problem can be solved by restricting $Q(z)$ to vary in a finite-dimensional subspace of the space of stable matrices. Moreover, to be able to satisfy the design objectives, usually the resulting subspace needs to be of very high dimension, resulting in a very high-order controller, even for low-order plants.

Recently, several researchers have attempted to formulate a state-space discrete-time control design problem as a finite-dimensional convex optimization problem, most notably Geromel et al. (Refs. 5, 6) and Kaminer et al. (Ref. 7). The key idea in these papers is to introduce a change of variables which replaces the search over the nonconvex space of the controller parameters by a search over a new parameter space which is convex. The optimization problem is solved in the new parameter space, and a feasible stabilizing controller is obtained by a reverse change of variables from the new parameter space to the space of controller parameters. The resulting optimization problem is nonsmooth; hence, nondifferentiable optimization techniques must be applied for a solution (Refs. 3, 8).

In this paper, the discrete-time control design problem is solved in the parameter space of the plant-state covariance matrices. The design problem formulated in terms of the state covariance matrix is convex, finite-dimensional, and can lead to low-order controllers. Moreover, it has an appealing physical interpretation (the covariance matrix) and engineering motivation. It is known that stability, performance, robustness, pole location, and many other closed-loop design objectives can be related directly to the covariance matrix (Refs. 9–13), thus providing a multiobjective flavor to the control design problem, which is absent from the traditional single-objective design techniques such as LQG or H^∞.

On the other hand, covariance control theory (Refs. 9, 10, 14, 15) provides a characterization of all assignable covariances and in addition a parametrization of all controllers which assign a particular assignable covariance. As a result, covariance control theory can be seen as a state-space parametrization of all stabilizing controllers of fixed order, resembling the Q-parametrization discussed above, but where now the parameter space is the space of covariance matrices. A noticeable advantage of this formulation of the problem over the frequency domain Q-parametrization is the finite dimensionality of the parameter space and the ability to fix the controller order to be equal or less than the order of the plant. By a slight abuse of language, the term covariance matrix will indicate the solution of the closed-loop Lyapunov equation, and it can have a stochastic as well as a deterministic interpretation (Ref. 16).

Following this philosophy, the covariance control design problem is based on these three steps:

(S1) formulate the desired control design objectives as constraints in the space of assignable covariance matrices;
(S2) use numerical techniques to obtain an assignable covariance which satisfy the desired objectives;
(S3) parametrize (in closed form) the set of all controllers which assign the desired covariance, computed in Step (S2); obtain a satisfactory one, according to some given criteria.

In this paper, the constraints of Step (S1) will be assignability constraints and multiple-output norm constraints. We will show that these constraints represent closed and convex sets of simple geometry in the space of symmetric matrices, hence providing a very appealing geometric interpretation to the covariance design problem. A desired covariance is one which lies in the intersection of these constrained sets. Our geometric approach of Step (S1) leads us to some additional techniques for the algorithmic implementation of Step (S2). We first note that the finite dimensionality and convexity of the parameter space in Step (S1) allows the possibility to use effective convex optimization techniques to obtain a solution of Step (S2). On the other hand, the simple geometric structure of the constraint sets in Step (S1) will give us the ability to derive an analytic expression for the projection operator onto each constraint set. Based on these results, alternating convex projection methods are proposed to solve Step (S2).

Alternating projection methods (Refs. 17–19) have been used successfully in statistical estimation and image restoration problems (see Refs.

20–23 and references therein). They provide iterative schemes for finding a feasible point in the intersection of a family of closed convex sets or for finding the minimum distance solution with respect to this intersection. The basic idea is that of a cyclic sequence of projections onto the constraint sets; hence, expressions for the projection operators onto each individual set are needed.

In this paper, we provide a geometric formulation of the following problems, and alternating projection-type techniques are proposed for a solution.

(P1) Covariance Feasibility Problem. We look for a covariance matrix to satisfy the constraints imposed in Step (S1). Any feasible solution which satisfies these constraints is a valid one.

(P2) Covariance Optimization Problem. We seek the covariance matrix to satisfy the constraints imposed in Step (S1), which is as close as possible to a given desired, but unassignable covariance matrix.

(P3) Covariance Suboptimization Problem. We look for a covariance matrix to satisfy the constraints of Step (S1), which is less than a prespecified distance from a given desired, but unassignable covariance matrix.

(P4) Infeasible Covariance Optimization Problem. We seek an assignable covariance which approximates as close as possible the desired design objectives. This corresponds to the case where the desired constraints of Step (S1) are inconsistent; this happens often in practice, since the designer does not know a priori if the desired specifications are achievable.

In Step (S3), from the set of all controllers which assign the feasible covariance obtained in Step (S2), we will choose a desired one according to some given criteria (e.g., to minimize the required active control effort). In addition, a technique to design low-order controllers is proposed by restricting the covariance matrix to satisfy a rank condition. Alternating projection techniques are suggested for a solution of this problem, although convergence of the method is not guaranteed in this case.

The results in this paper follow the corresponding results for continuous-time covariance control developed in Ref. 24. In addition to the discrete-time counterparts of the results in Ref. 24, the following problems are discussed in conjunction with the alternating convex projection technique: covariance suboptimization problem, block output variance constrained problem, and reduced-order controller design problem.

Section 2 summarizes the necessary mathematical background from discrete-time covariance control theory. Section 3 shows the convexity of the assignability and performance constraint sets and formulates the covariance design problems to be solved. Section 4 presents the alternating projection algorithms. Section 5 provides the analytic expressions for the required projection operators and shows how to use these algorithms to solve the problems in Section 3. Section 6 discusses the problem of reduced-order dynamic controller design. Section 7 provides a numerical example; some conclusions are offered in Section 8.

We denote by the superscript T the transpose of a real matrix, by the superscript $+$ the Moore–Penrose generalized inverse of a matrix, and by the superscript * a feasible or optimal solution. The Kronecker product between two matrices $A=(a_{ij})$ and B is defined as $A\otimes B=(a_{ij}B)$, that is, the block matrix with submatrices $a_{ij}B$. We denote by vec(A) the operator which stacks the columns of a matrix A one underneath the other, in one column vector. The square root of a positive-semidefinite matrix X is the unique positive-semidefinite matrix $X^{1/2}$ that satisfies

$$X^{1/2}X^{1/2}=X.$$

The square factor of a positive-semidefinite matrix X is any square matrix T that satisfies

$$TT^T=X.$$

The Frobenious norm of a matrix X is

$$\|X\|=[\mathrm{tr}(XX^T)]^{1/2},$$

where $\mathrm{tr}(\cdot)$ denotes the trace operator.

2. Preliminaries

Both static state feedback and dynamic feedback covariance control problems are examined. It is shown that both cases result in similar mathematical problems. We first examine the static state feedback problem.

Consider a linear, time-invariant, discrete-time dynamic system with the following state-space representation:

$$x_p(k+1)=A_p x_p(k)+B_p u(k)+D_p w_p(k), \tag{1a}$$

$$y(k)=C_p x_p(k)+H_p w_p(k), \tag{1b}$$

where $x_p(k)\in\mathbb{R}^{n_x}$ is the plant-state vector and $y(k)\in\mathbb{R}^{n_y}$ is the vector of outputs whose performances are of interest. We assume that (A_p, D_p) is controllable (A_p, B_p) is stabilizable, and $\mathrm{Range}(B_p)\subset\mathrm{Range}(D_p)$. The last condition is always satisfied if the actuators are disturbance sources. In a stochastic interpretation of the system (1), $w_p(k) \in \mathbb{R}^{n_w}$ is a zero-mean white-noise process with covariance $W_p > 0$. We seek a state feedback control law

$$u(k) = Kx_p(k) \tag{2}$$

to satisfy desired closed-loop stability and performance objectives.

Combining (1) and (2), the closed-loop system obeys

$$x_p(k+1) = (A_p + B_p K)x_p(k) + D_p w_p(k), \tag{3}$$

and the closed-loop state covariance matrix

$$X_p \triangleq \lim_{k \to \infty} \mathscr{E}\{x_p(k)x_p^T(k)\} \tag{4}$$

satisfies the following discrete-time Lyapunov equation:

$$X_p = (A_p + B_p K)X_p(A_p + B_p K)^T + D_p W D_p^T. \tag{5}$$

The controllability of (A_p, D_p) and our assumptions on the relation between D_p and B_p imply that $(A_p + B_p K, D_p)$ is controllable. Hence, Lyapunov stability theory guarantees that the closed-loop system is asymptotically stable (i.e., every eigenvalue of $A_p + B_p K$ has magnitude less than unity) if and only if $X_p > 0$. Hence, we are interested only in positive-definite state covariances which satisfy Eq. (5).

Definition 2.1. A closed-loop state covariance $X_p > 0$ is assignable to the closed-loop system if X_p satisfies Eq. (5) for some state feedback controller K.

The concepts of covariance control theory for discrete-time state feedback systems originated in Ref. 10 to provide a multiobjective flavor to the discrete-time control design problem. The mathematical objective of covariance control theory is to provide necessary and sufficient conditions for a covariance matrix to be assignable and a parametrization of all control gains which assign a particular covariance. The following results are from Ref. 10.

Theorem 2.1. The set of all assignable state covariances X_p is parametrized by

$$(I - B_p B_p^+)(X_p - A_p X_p A_p^T - D_p W_p D_p^T)(I - B_p B_p^+) = 0, \tag{6a}$$

$$X_p > 0, \tag{6b}$$

$$X_p \geq D_p W_p D_p^T. \tag{6c}$$

Theorem 2.2. If a state covariance X_p is assignable, then all state feedback gains which assign X_p to the discrete-time system are parametrized by

$$K = B_p^+ \left\{ (X_p - D_p W_p D_p^T)^{1/2} F_1 \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} F_2^T T^{-1} - A_p \right\} + (I - B_p^+ B_p) Z, \tag{7a}$$

where Z is an arbitrary matrix, U is an arbitrary orthogonal matrix, T is a square factor of X, and F_1, F_2 are defined from the singular value decompositions

$$(I - B_p B_p^+)(X_p - D_p W_p D_p^T)^{1/2} = E \Sigma F_1^T, \tag{7b}$$

$$(I - B_p B_p^+) A T = E \Sigma F_2^T, \tag{7c}$$

where E, F_1, F_2 are orthogonal matrices.

Next, we examine the discrete-time dynamic controller case with measurement noise. We consider the linear discrete-time dynamic system (1) with the measurement equation

$$z(k) = M_p x_p(k) + v(k), \tag{8}$$

where $z(k) \in \mathbb{R}^{n_z}$ is the vector of noisy measurements and $v(k) \in \mathbb{R}^{n_z}$ is a zero-mean white-noise process with covariance $V > 0$. We assume that (A_p, M_p) is a detectable pair. We wish to find a discrete-time dynamic controller with state space representation

$$x_c(k+1) = A_c x_c(k) + B_c z(k), \tag{9a}$$

$$u(k) = C_c x_c(k) + D_c z(k), \tag{9b}$$

where $x_c(k) \in \mathbb{R}^{n_c}$. Combining (1), (8), (9), the closed-loop system can be expressed in the following form:

$$x(k+1) = (A + BGM) x(k) + (D + BGJ) w(k), \tag{10}$$

where

$$x \triangleq \begin{bmatrix} x_p \\ x_c \end{bmatrix}, \qquad w \triangleq \begin{bmatrix} w_p \\ v \end{bmatrix}, \tag{11}$$

and

$$A \triangleq \begin{bmatrix} A_p & 0 \\ 0 & 0 \end{bmatrix}, \qquad B \triangleq \begin{bmatrix} B_p & 0 \\ 0 & I_{n_c} \end{bmatrix}, \qquad M \triangleq \begin{bmatrix} M_p & 0 \\ 0 & I_{n_c} \end{bmatrix}, \tag{12a}$$

$$D \triangleq \begin{bmatrix} D_p & 0 \\ 0 & 0 \end{bmatrix}, \qquad J \triangleq \begin{bmatrix} 0 & I_{n_c} \\ 0 & 0 \end{bmatrix}, \qquad G \triangleq \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}. \tag{12b}$$

The closed-loop state covariance

$$X \triangleq \lim_{k\to\infty} \mathscr{E}\{x(k)x^T(k)\} \tag{13}$$

satisfies the discrete Lyapunov equation

$$X = (A+BGM)X(A+BGM)^T + (D+BGJ)W(D+BGJ)^T, \tag{14}$$

where the noise covariance matrix W and the state covariance matrix X are partitioned as follows:

$$W \triangleq \begin{bmatrix} W_p & 0 \\ 0 & V \end{bmatrix}, \qquad X = \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^T & X_c \end{bmatrix}. \tag{15}$$

Closed-loop stability is equivalent to the condition $X > 0$. The following definition generalizes Definition 2.1 for the dynamic controller case.

Definition 2.2. A closed-loop state covariance $X > 0$ is assignable to the closed-loop system if X satisfies (14) for some G.

Since the closed-loop output performance is related only to the plant covariance matrix X_p, for our purposes we are interested only in the assignability of X_p. Hence, we need the following definition.

Definition 2.3. A plant-state covariance $X_p > 0$ is assignable if there exist X_c and X_{pc} in (14) such that $X > 0$ is assignable.

Since (A_p, D_p) is controllable, it can be shown that $(A+BGM, D+BGJ)$ is controllable provided the controller is minimal, i.e., (A_c, B_c, C_c) is controllable and observable. Therefore, by the Lyapunov theory, $A+BGM$ is asymptotically stable if and only if $X > 0$. Hence, $X_p > 0$ and assignability guarantees stabilizability of the system. The following results are from Ref. 15.

Theorem 2.3. A matrix $X_p > 0$ is an assignable plant-state covariance if and only if it satisfies

$$(I - B_pB_p^+)(X_p - A_pX_pA_p^T - D_pW_pD_p^T)(I - B_pB_p^+) = 0, \tag{16a}$$

$$X_p \geq P, \tag{16b}$$

where P is the unique positive-definite solution of the following discrete-time algebraic Riccati equation:

$$P = A_pPA_p^T - A_pPM_p^T(M_pPM_p^T + V)^{-1}M_pPA_p^T + D_pW_pD_p^T. \tag{17}$$

Theorem 2.4. Let X_p be an assignable plant-state covariance. Then, a closed-loop state covariance X defined by (15) is assignable if and only if X_c, X_{pc} satisfy

$$\bar{X}_p \triangleq X_p - X_{pc} X_c^{-1} X_{pc}^T > 0, \tag{18a}$$

and $\bar{X}_p$ is the positive-definite solution of the following Riccati equation:

$$\begin{aligned}\bar{X}_p &= A_p \bar{X}_p A_p^T - A_p \bar{X}_p M_p^T (M_p \bar{X}_p M_p^T + V)^{-1} M_p \bar{X}_p A_p^T \\ &\quad + D_p W_p D_p^T + \bar{L}\bar{L}^T,\end{aligned} \tag{18b}$$

for some $\bar{L} \in \mathbb{R}^{n_x \times n_z}$. The set of all controllers which assign X to the closed-loop system is parametrized by

$$\begin{aligned}G &= B^+ \left\{ L F_1 \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} F_2^T - (AXM^T + DWJ^T)\Gamma^{-T} \right\} \Gamma^{-1} \\ &\quad + (I - B^+ B) Z.\end{aligned} \tag{18c}$$

Here, U is arbitrary row orthogonal; Z is arbitrary; L, Γ are defined by

$$\begin{aligned}LL^T &= X - AXA^T - DWD^T \\ &\quad + (AXM^T + DWJ^T)(MXM^T + JWJ^T)^{-1} \\ &\quad \times (AXM^T + DWJ^T)^T, \qquad L \in \mathbb{R}^{(n_x + n_c) \times (n_z + n_c)},\end{aligned} \tag{18d}$$

$$\Gamma\Gamma^T = MXM^T + JWT^T, \qquad \Gamma \in \mathbb{R}^{(n_z + n_c) \times (n_z + n_c)}; \tag{18e}$$

and F_1, F_2 are defined from the singular-value decompositions

$$(I - BB^+) L = E_1 \Sigma F_1^T, \tag{18f}$$

$$(I - BB^+)(AXM^T + DWJ^T)\Gamma^{-T} = E_1 \Sigma F_2^T. \tag{18g}$$

This result parametrizes all assignable closed-loop state covariances and all controllers which assign a state covariance to the closed-loop system.

Theorems 2.1 and 2.3 indicate that the state feedback and full-order plant state covariance assignability problems have similar structure, namely, that of finding a matrix that satisfies a linear matrix equation [(6a) or (16a)] and a matrix positivity constraint [(6b) or (16b)]. On the other hand, several closed-loop performance and robustness requirements can be expressed as constraints on the plant covariance matrix X_p. The plant covariance design problem is to find an assignable plant state covariance matrix which satisfies desired performance and robustness constraints. Therefore, covariance control theory allows the control design problem to be formulated in terms of an assignable plant-state covariance $X_p > 0$, eliminating the controller parameters from the design problem. When a desired assignable plant-state covariance is obtained, covariance control theory provides directly the analytical expressions for all the controllers which assign this covariance to the closed-loop system.

In the next section, we provide a geometric formulation of the assignability and the output performance constraints in terms of the plant-state covariance matrix X_p.

3. Convex Constraint Formulation of the Covariance Design Problem

Consider the $n_x(n_x+1)/2$-dimensional vector space $\mathscr{S}_{n_x}$ of $n_x \times n_x$ real symmetric matrices. We characterize the assignability and output performance objectives as closed convex constraints sets in $\mathscr{S}_{n_x}$. This approach provides a new geometric interpretation of the covariance design problem.

3.1. Assignability Constraints on $\mathscr{S}_{n_x}$. Consider the following constraint sets in $\mathscr{S}_{n_x}$:

$$\mathscr{A} \triangleq \{X_p \in \mathscr{S}_{n_x}: (I - B_pB_p^+)(X_p - A_pX_pA_p^T - D_pW_pD_p^T)(I - B_pB_p^+) = 0\}, \tag{19}$$

and

$$\mathscr{P}_1 \triangleq \{X_p \in \mathscr{S}_{n_x}: X_p \geq P\}, \tag{20a}$$

$$\mathscr{P}_2 \triangleq \{X_p \in \mathscr{S}_{n_x}: X_p > 0\}. \tag{20b}$$

In the definition of $\mathscr{P}_1$, we require that

$$P = D_pW_pD_p^T \tag{21a}$$

for the state feedback problem presented in Theorem 2.1, and P to be the positive-definite solution of the following Riccati equation:

$$P = A_pPA_p^T - A_pPM_p^T(M_pPM_p^T + V)^{-1}M_pPA_p^T + D_pW_pD_p^T \tag{21b}$$

for the dynamic controller case presented in Theorem 2.3. Note that $\mathscr{A}$ is an affine manifold (a translation of a linear subspace) in $\mathscr{S}_{n_x}$ and $\mathscr{P}_1$, $\mathscr{P}_2$ are convex cones with vertices at P and 0, respectively; i.e., if $P + X_p$ is an element of $\mathscr{P}_1$, then also $P + \rho X_p$ is an element of $\mathscr{P}_1$, for any scalar $\rho \geq 0$, and similarly for $\mathscr{P}_2$. The set $\mathscr{P} = \mathscr{P}_1 \cap \mathscr{P}_2$ is also a convex cone as an intersection of two convex cones (Ref. 25). Obviously, $\mathscr{P} = \mathscr{P}_1$ for the dynamic controller case, since in that case $\mathscr{P}_1 \subset \mathscr{P}_2$. The plant-state covariance assignability conditions of Theorem 2.1 or Theorem 2.3 can be formulated equivalently as follows.

Proposition 3.1. A matrix $X_p \in \mathscr{S}_{n_x}$ is an assignable plant-state covariance if and only if

$$X_p \in \mathscr{A} \cap \mathscr{P}, \qquad \mathscr{P} = \mathscr{P}_1 \cap \mathscr{P}_2, \tag{22}$$

where P is provided by (21a) or (21b) for the static state feedback and the dynamic controller case, respectively.

Hence, the set of assignable plant covariances is precisely the intersection of the affine manifold $\mathscr{A}$ with the convex cone $\mathscr{P}$. Note that the stabilizability of the system is equivalent to the condition that the intersection $\mathscr{A} \cap \mathscr{P}$ is nonempty.

Since we will require the constraint sets to be closed sets, we replace $\mathscr{P}_2$ in (20b) by

$$\mathscr{P}_{2,\epsilon} \triangleq \{X_p \in \mathscr{S}_{n_x} : X_p \geq \epsilon I_{n_x}\}, \tag{23}$$

for an arbitrarily small $\epsilon > 0$, and we define the closed convex cone $\mathscr{P}_\epsilon \triangleq \mathscr{P}_1 \cap \mathscr{P}_{2\epsilon}$. We call any matrix $X_p \in \mathscr{A} \cap \mathscr{P}_\epsilon$ an ϵ-assignable plant covariance. Obviously, ϵ-assignability implies assignability; in addition, when $\epsilon \to 0$ with $\epsilon > 0$, then the set $\mathscr{A} \cap \mathscr{P}_\epsilon$ tends to the set $\mathscr{A} \cap \mathscr{P}$. Hence, in the following, we will not distinguish between ϵ-assignability and assignability.

The following result shows a very important property of the set of all assignable plant-state covariances.

Proposition 3.2. The set of all assignable plant-state covariances $\mathscr{A} \cap \mathscr{P}_\epsilon$ is a closed and convex subset of $\mathscr{S}_{n_x}$.

The proof of this result is a direct consequence of the fact that the intersection of closed convex sets is closed and convex.

3.2. Performance Constraints on $\mathscr{S}_{n_x}$. Suppose that the system output vector $y(k) \in \mathbb{R}^{n_y}$ consists of the group of m vectors $y_1(k) \in \mathbb{R}^{n_{y1}}$, $y_2(k) \in \mathbb{R}^{n_{y2}}, \ldots, y_m(k) \in \mathbb{R}^{n_{ym}}$; i.e.,

$$y^T = [y_1^T, y_2^T, \ldots, y_m^T], \tag{24}$$

where $\sum_{k=1}^{n_{yk}} = n_y$. Variance performance requirements on the system outputs correspond to matrix inequality constraints on the block diagonal elements of the output covariance matrix defined by

$$Y \triangleq \lim_{k \to \infty} \mathscr{E}\{y(k) y^T(k)\} = C_p X_p C_p^T + H_p W_p H_p^T. \tag{25}$$

Let

$$C_p^T = [C_{p1}^T, C_{p2}^T, \ldots, C_{pm}^T], \tag{26a}$$

$$H_p^T = [H_{p1}^T, H_{p2}^T, \ldots, H_{pm}^T] \tag{26b}$$

be the block decomposition of C_p and H_p according to the output groups (24). Then the ith, $i=1, \ldots, m$, block variance constraint has the following form:

$$\lim_{k\to\infty} \mathscr{E}\{y_i(k)y_i^T(k)\} = C_{pi}X_pC_{pi}^T + H_{pi}W_pH_{pi}^T \leq Y_i, \tag{27}$$

where $Y_i \in \mathbb{R}^{n_{yi}\times n_{yi}}$ is a given bounding matrix. Define the following subsets of $\mathscr{S}_{n_x}$:

$$\mathcal{O}_i \triangleq \{X_p \in \mathscr{S}_{n_x}: C_{pi}X_pC_{pi}^T \leq Y_i - H_{pi}W_pH_{pi}^T\}, \qquad i=1, \ldots, m. \tag{28}$$

Then, the requirement that the system satisfy the ith block variance constraint is equivalent to the constraint

$$X_p \in \mathcal{O}_i. \tag{29}$$

The following result can be proved easily using the definition of convexity.

Proposition 3.3. The set $\mathcal{O}_i$ as defined by (28) is a closed and convex subset of $\mathscr{S}_{n_x}$. Specifically, $\mathcal{O}_i$ is a closed convex cone with vertex $Y_i - H_{pi}W_pH_{pi}^T$.

The subset of $\mathscr{S}_{n_x}$ where all the desired variance constraints are satisfied is the intersection

$$\mathcal{O} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m. \tag{30}$$

Proposition 3.4. The set $\mathcal{O}$ is a closed and convex subset of $\mathscr{S}_{n_x}$.

Note that $\mathcal{O}$ is a closed convex set as an intersection of a finite family of closed convex sets.

As a special case, suppose that

$$n_{yi} = 1, \qquad i=1, \ldots, m;$$

i.e., C_{pi} and H_{pi} are the rows of the matrices C_p and H_p, respectively. In this case, the output variance constraint problem

$$[Y]_{ii} = [C_pX_pC_p^T + H_pW_pH_p^T]_{ii} \leq \sigma_i, \qquad i=1, \ldots, n_y, \tag{31}$$

is obtained (Refs. 10, 14), where $[Y]_{ii}$ is the ith diagonal element of the output covariance matrix Y and σ_i are given bounds.

Next, we consider a constraint on the weighted closed-loop system output cost defined as

$$J_y \triangleq \lim_{k\to\infty} \mathscr{E}\{y^T(k)Qy(k)\}, \tag{32}$$

where $Q \geq 0$ is a constant weighting matrix. It can be verified easily that

$$J_y = \operatorname{tr}(X_p C_p^T Q C_p). \tag{33}$$

Suppose that the desired output cost constraint is

$$J_y \leq \gamma, \tag{34}$$

where $\gamma > 0$ is a given scalar. By defining the set

$$\mathscr{C} \triangleq \{X_p \in \mathscr{S}_{n_x} : \operatorname{tr}(X_p C_p^T Q C_p) \leq \gamma\}, \tag{35}$$

the constraint (34) is equivalent to requiring

$$X_p \in \mathscr{C}. \tag{36}$$

The following result is easy to verify.

Proposition 3.5. The set $\mathscr{C}$ is a closed and convex subset of $\mathscr{S}_{n_x}$.

Note that $\mathscr{C}$ is a closed half-space in $\mathscr{S}_{n_x}$.

3.3. Covariance Feasibility Problem. The covariance control design problem subject to output variance and output cost constraints of the form (27) and (34), respectively, can be formulated as the following feasibility problem:

(P1) Find a matrix $X_p^* \in \mathscr{S}_{n_x}$ such that

$$X_p^* \in \mathscr{A} \cap \mathscr{P}_\epsilon \cap \mathscr{O} \cap \mathscr{C}, \tag{37}$$

or determine that none exists. Note that the set $\mathscr{A} \cap \mathscr{P}_\epsilon \cap \mathscr{O} \cap \mathscr{C}$ is closed and convex as an intersection of a finite number of closed and convex sets. Although under stabilizability assumptions the set of assignable plant covariances is nonempty, there is no guarantee that there exists a solution to the covariance feasibility problem, since the set of desirable plant covariances on the right-hand side of (37) might be empty. This might often be the case in practice, since the desired performance constraints might not be achievable. Hence, it is important that the methodology proposed to solve (37) can provide information about the existence of a solution.

3.4. Covariance Optimization Problem. The covariance control design problem can be formulated alternatively as an optimization problem in

$\mathscr{S}_{n_x}$. Suppose that $X_p > 0$ denotes a desirable plant-state covariance constructed to represent some desired system performance. Note that X_p might not be assignable. Hence, we are interested to find an assignable covariance X_p^* which satisfies the desired performance objectives (27) and (34) and is as close as possible to X_p. Thus, we look for a solution to the following problem:

(P2) Given $X_p \in \mathscr{S}_{n_x}$, find a matrix $X_p^* \in \mathscr{S}_{n_x}$ to solve

$$\min \|X_p - X_p^*\|, \qquad \text{s.t. } X_p^* \in \mathscr{A} \cap \mathscr{P}_\epsilon \cap \mathscr{O} \cap \mathscr{C}. \tag{38}$$

Note that, if $\mathscr{A} \cap \mathscr{P}_\epsilon \cap \mathscr{O} \cap \mathscr{C}$ is nonempty, then the covariance optimization problem has a unique solution.

3.5. Covariance Suboptimization Problem. Often, it is desirable to approximate the optimization problem (38) with the following suboptimization problem:

(P3) Given a matrix $X_p \in \mathscr{S}_{n_x}$ and a scalar $\delta > 0$, find a matrix $X_p^* \in \mathscr{S}_{n_x}$ such that

$$X_p^* \in \mathscr{A} \cap \mathscr{P}_\epsilon \cap \mathscr{O} \cap \mathscr{C}, \tag{39}$$

$$\|X_p - X_p^*\| \leq \delta. \tag{40}$$

It is apparent that, by defining the closed ball around X_p,

$$\mathscr{B} = \{Z \in \mathscr{S}_{n_x} : \|Z - X_p\| \leq \delta\}, \tag{41}$$

then the suboptimization problem (39)–(40) is equivalent to the following feasibility problem:

Find a matrix $X_p^* \in \mathscr{S}_{n_x}$ such that

$$X_p^* \in \mathscr{A} \cap \mathscr{P}_\epsilon \cap \mathscr{O} \cap \mathscr{C} \cap \mathscr{B}. \tag{42}$$

3.6. Infeasible Covariance Optimization Problem. In the case of inconsistent constraints, i.e., when $\mathscr{A} \cap \mathscr{P}_\epsilon \cap \mathscr{O} \cap \mathscr{C}$ is empty, it is desirable to find an assignable covariance which approximates closely the desired but unachievable performance objectives (27) or (34). Note that the stabilizability of the plant is equivalent to $\mathscr{A} \cap \mathscr{P}_\epsilon \neq \varnothing$. Also, $\mathscr{O} \cap \mathscr{C} \neq \varnothing$ since, for example, the zero matrix is an element of this set. Hence, in this case we are motivated to solve the following problem:

(P4) Find a matrix $X_p^* \in \mathscr{S}_{n_x}$ to solve

$$\min \operatorname{dist}(X_p^*, \mathscr{O} \cap \mathscr{C}), \qquad \text{s.t. } X_p^* \in \mathscr{A} \cap \mathscr{P}_\epsilon; \tag{43}$$

i.e., X_p^* is the assignable plant covariance which minimizes the distance from the set of desired performance constraints.

In the next section, a method of alternating projections is proposed to solve the above covariance design problems, taking advantage of the simple geometric structure of the problems.

4. Alternating Convex Projections Method

Consider a finite-dimensional Hilbert space $\mathscr{H}$, and let $\|\cdot\|$ be the norm induced by the inner product $\langle\cdot,\cdot\rangle$. Let $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ be a family of closed convex sets in $\mathscr{H}$, and define $\mathscr{C} \triangleq \mathscr{C}_1 \cap \mathscr{C}_2 \cap \cdots \cap \mathscr{C}_n$. We seek to solve the following problems.

(Q1) Feasibility Problem. Suppose that $\mathscr{C}$ is nonempty. Find a vector $\mathscr{X} \in \mathscr{H}$ such that

$$\mathscr{X} \in \mathscr{C}. \tag{44}$$

(Q2) Optimization Problem. Suppose that $\mathscr{C}$ is nonempty. Find a vector $\mathscr{X} \in \mathscr{H}$ to solve

$$\min \|\mathscr{X} - \mathscr{X}_0\|, \qquad \text{s.t. } \mathscr{X} \in \mathscr{C}. \tag{45}$$

(Q3) Infeasible Optimization Problem. Given two disjoint closed convex sets $\mathscr{C}_1$ and $\mathscr{C}_2$ in $\mathscr{H}$, find a vector $\mathscr{X} \in \mathscr{C}_2$ to solve

$$\min \operatorname{dist}(\mathscr{X}, \mathscr{C}_1), \tag{46}$$

where $\operatorname{dist}(\mathscr{X}, \mathscr{C}_1)$ is the distance between the vector $\mathscr{X}$ and $\mathscr{C}_1$ defined by

$$\operatorname{dist}(\mathscr{X}, \mathscr{C}_1) = \min \|\mathscr{X} - \hat{\mathscr{X}}\|, \qquad \text{such that } \hat{\mathscr{X}} \in \mathscr{C}_1. \tag{47}$$

The feasibility problem (Q1) seeks any point in the intersection of $\mathscr{C}_i$, $i = 1, \ldots, n$; the optimization problem (Q2) seeks the point in the intersection which is closest to a given point in $\mathscr{H}$; the infeasible optimization problem (Q3) considers the case where $\mathscr{C}_1 \cap \mathscr{C}_2 = \varnothing$ and looks for a point in $\mathscr{C}_2$ which is closest to the set $\mathscr{C}_1$.

Note that the optimization problem (Q2) has a unique solution according to the following well-known result (e.g., see Ref. 26).

Theorem 4.1. Let $\mathscr{K} \neq \varnothing$ be a closed and convex set in $\mathscr{H}$ and $\mathscr{X}_0 \in \mathscr{H}$. Then, there exists a unique vector $\mathscr{X}^* \in \mathscr{K}$ which is closest to $\mathscr{X}_0$; i.e., $\mathscr{X}^*$

satisfies

$$\|\mathscr{X}^* - \mathscr{X}_0\| \leq \|\hat{\mathscr{X}} - \mathscr{X}_0\|, \qquad \text{for any } \hat{\mathscr{X}} \in \mathscr{K}. \tag{48}$$

Furthermore, $\mathscr{X}^*$ is uniquely characterized by the condition

$$\langle \mathscr{X}^* - \mathscr{X}_0, \mathscr{X}^* - \hat{\mathscr{X}} \rangle \leq 0, \qquad \text{for any } \hat{\mathscr{X}} \in \mathscr{K}. \tag{49}$$

We will need the following definitions: A vector $\mathscr{X}^* \in \mathscr{K}$ which solves (48) is called the projection of $\mathscr{X}$ on $\mathscr{K}$ and is denoted by $\mathscr{X}^* \triangleq P_{\mathscr{K}}(\mathscr{X})$. Note that the projection operator $P_{\mathscr{K}}$ is a linear operator if and only if $\mathscr{K}$ is a subspace. A fixed point of an operator $\Phi: \mathscr{H} \to \mathscr{H}$ is any vector $\mathscr{X} \in \mathscr{H}$ such that $\Phi(\mathscr{X}) = \mathscr{X}$. It can be observed easily that a vector $\mathscr{X}$ is a fixed point of the projection operator $P_{\mathscr{K}}$ if and only if $\mathscr{X} \in \mathscr{K}$.

A simple iterative scheme to solve the feasibility problem (Q1), the optimization problem (Q2), and the infeasible optimization problem (Q3) is provided by the method of alternating convex projections (Refs. 17–19). This method consists of cyclic successive projections onto each convex set $\mathscr{C}_i$, $i = 1, \ldots, n$. The standard method (Ref. 17) solves problems (Q1) and (Q3); a modification of the method can solve problem (Q2) (Ref. 19). These methods have been applied successfully in statistics, signal restoration, image reconstruction problems, and other fields of data analysis (Refs. 20–22). To implement the alternating convex projections method, an expression for the projection $P_{\mathscr{C}_i}(\mathscr{X})$ of $\mathscr{X}$ on each convex set $\mathscr{C}_i$ is needed; hence, the corresponding convex sets should be of simple structure so that such an expression can be derived. The convex projections method can be described as follows.

(A1) Alternating Projection Algorithm for the Feasibility Problem. Let $\mathscr{X}_0 \in \mathscr{H}$, and define:

(1st cycle) $$\mathscr{X}_1 = P_{\mathscr{C}_1} \mathscr{X}_0, \tag{50a}$$

$$\mathscr{X}_2 = P_{\mathscr{C}_2} \mathscr{X}_1, \tag{50b}$$

$$\vdots$$

$$\mathscr{X}_n = P_{\mathscr{C}_n} \mathscr{X}_{n-1}; \tag{50c}$$

(2nd cycle) $$\mathscr{X}_{n+1} = P_{\mathscr{C}_1} \mathscr{X}_n, \tag{50d}$$

$$\vdots$$

$$\mathscr{X}_{2n} = P_{\mathscr{C}_n} \mathscr{X}_{2n-1}; \tag{50e}$$

(3rd cycle) $$\mathscr{X}_{2n+1} = P_{\mathscr{C}_1} \mathscr{X}_{2n}, \tag{50f}$$

$$\vdots$$

$$\mathscr{X}_{3n} = P_{\mathscr{C}_n} \mathscr{X}_{3n-1}; \tag{50g}$$

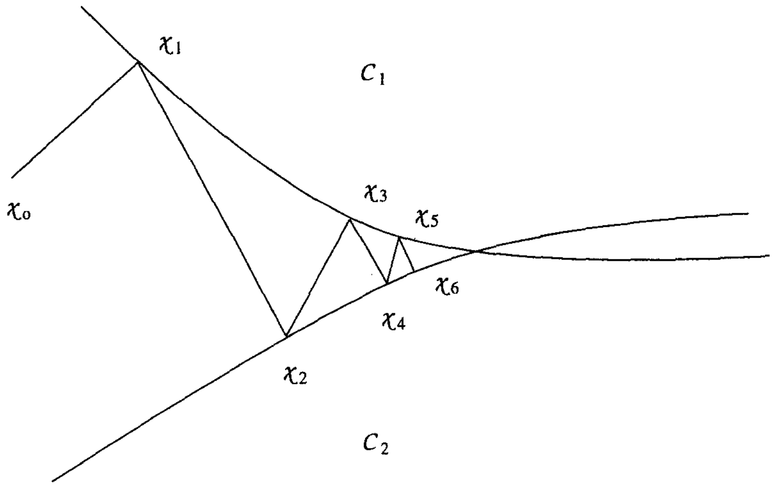

Fig. 1. Alternating projection algorithm.

and so on. Hence, the sequence of vectors $\{\mathscr{X}_k\}_{k=0}^{\infty}$ is formed by successive projections in a cyclic manner on the convex sets $\mathscr{C}_i$, $i=1,\ldots,n$. A sketch of the method for $n=2$ is given in Fig. 1. The following result, proved in Ref. 18, points out the global convergence property of the sequence $\{\mathscr{X}_k\}_{k=0}^{\infty}$.

Theorem 4.2. The sequence $\{\mathscr{X}_k\}_{k=0}^{\infty}$ generated by (50) converges to a vector $\mathscr{X}$ in $\mathscr{C} \triangleq \mathscr{C}_1 \cap \mathscr{C}_2 \cap \cdots \cap \mathscr{C}_n$ for any initial vector $\mathscr{X}_0 \in \mathscr{H}$.

It can be shown that the intersection $\mathscr{C}$ is exactly the set of fixed points of the operator $\Phi \triangleq P_{\mathscr{C}_n} P_{\mathscr{C}_{n-1}} \cdots P_{\mathscr{C}_1}$, which is formed from the composition of the projections. This is a generalization of the well-known fixed-point theorem of contractive operators (Ref. 26). Therefore, Theorem 4.2 provides an iterative scheme for the solution of the feasibility problem (Q1).

A simple example can illustrate that in general the limit of the sequence $\{\mathscr{X}_k\}_{k=0}^{\infty}$ of Theorem 4.2 is not the projection of $\mathscr{X}_0$ onto the intersection $\mathscr{C}$; i.e., Algorithm (A1) does not solve the optimization problem (42); e.g., consider $\mathscr{C}_1$ to be a disc in $\mathbb{R}^2$ and $\mathscr{C}_2$ to be a diameter of the disc. Nevertheless, a simple modification of Algorithm (A1), provided in Ref. 19, can solve the optimization problem (Q2).

(A2) Alternating Projection Algorithm for the Optimization Problem. Let $\mathscr{X}_0 \in \mathscr{H}$, and define:

$$\text{(1st cycle)} \quad \mathscr{X}_1 = P_{\mathscr{C}_1}\mathscr{X}_0, \qquad \mathscr{Z}_1 = \mathscr{X}_1 - \mathscr{X}_0, \tag{51a}$$

$$\mathscr{X}_2 = P_{\mathscr{C}_2}\mathscr{X}_1, \qquad \mathscr{Z}_2 = \mathscr{X}_2 - \mathscr{X}_1, \tag{51b}$$

$$\vdots \qquad\qquad \vdots$$

$$\mathscr{X}_n = P_{\mathscr{C}_n}\mathscr{X}_{n-1}, \qquad \mathscr{Z}_n = \mathscr{X}_n - \mathscr{X}_{n-1}; \tag{51c}$$

$$\text{(2nd cycle)} \quad \mathscr{X}_{n+1} = P_{\mathscr{C}_1}(\mathscr{X}_n - \mathscr{Z}_1), \qquad \mathscr{Z}_{n+1} = \mathscr{Z}_1 + \mathscr{X}_{n+1} - \mathscr{X}_n, \tag{51d}$$

$$\vdots \qquad\qquad \vdots$$

$$\mathscr{X}_{2n} = P_{\mathscr{C}_n}(\mathscr{X}_{2n-1} - \mathscr{Z}_n), \qquad \mathscr{Z}_{2n} = \mathscr{Z}_n + \mathscr{X}_{2n} - \mathscr{X}_{2n-1}; \tag{51e}$$

$$\text{(3rd cycle)} \quad \mathscr{X}_{2n+1} = P_{\mathscr{C}_1}(\mathscr{X}_{2n} - \mathscr{Z}_{n+1}), \qquad \mathscr{Z}_{2n+1} = \mathscr{Z}_{n+1} + \mathscr{X}_{2n+1} - \mathscr{X}_{2n}, \tag{51f}$$

$$\vdots \qquad\qquad \vdots$$

$$\mathscr{X}_{3n} = P_{\mathscr{C}_n}(\mathscr{X}_{3n-1} - \mathscr{Z}_{2n}), \qquad \mathscr{Z}_{3n} = \mathscr{Z}_{2n} + \mathscr{X}_{3n} - \mathscr{X}_{3n-1}; \tag{51g}$$

and so on. Note that, in this modified projection algorithm, in each step j the increment $\mathscr{Z}_{j-n}$ is removed before projecting on the corresponding convex set. This forces the algorithm to converge to the solution $\mathscr{X}^*$ of the optimization problem (Q2). The next theorem guarantees this property (Ref. 19).

Theorem 4.3. The sequence $\{\mathscr{X}_k\}_{k=0}^{\infty}$ generated by (51) converges to the projection $\mathscr{X}^* \triangleq P_{\mathscr{C}}(\mathscr{X}_0)$; that is, $\mathscr{X}^*$ satisfies

$$\|\mathscr{X}_0 - \mathscr{X}^*\| \leq \|\mathscr{X}_0 - \hat{\mathscr{X}}\|, \qquad \forall \hat{\mathscr{X}} \in \mathscr{C}_1 \cap \mathscr{C}_2 \cap \cdots \cap \mathscr{C}_n. \tag{52}$$

The feasibility and optimization alternating projection algorithms (A1) and (A2) can be implemented simply; usually, the amount of calculations in one iteration is very small. However, the methods may suffer from slow convergence. It can be shown (Ref. 18) that the alternating projection method (A1) has a geometric rate of convergence, but this sometimes can result in slow convergence since the convergence ratio might be close to one (for example, when $\mathscr{C}_1$ and $\mathscr{C}_2$ are hyperplanes with a very small angle between them). One technique to accelerate the convergence is examined next (Ref. 18). For simplicity, we will examine the case $n = 2$, but the method can be generalized to any finite number of closed convex sets.

(A3) Directional Alternating Projection Algorithm. Let $\mathscr{X}_0 \in \mathscr{H}$ be given. Define the following sequence of vectors:

$$\text{(1st cycle)} \quad \mathscr{X}_1 = P_{\mathscr{C}_1}\mathscr{X}_0, \qquad \mathscr{X}_2 = P_{\mathscr{C}_2}\mathscr{X}_1, \qquad \mathscr{X}_3 = P_{\mathscr{C}_1}\mathscr{X}_2, \tag{53a}$$

$$\mathscr{X}_4 = \mathscr{X}_1 + \lambda_1(\mathscr{X}_3 - \mathscr{X}_1),$$

$$\lambda_1 = \|\mathscr{X}_1 - \mathscr{X}_2\|^2 / \langle \mathscr{X}_1 - \mathscr{X}_3, \mathscr{X}_1 - \mathscr{X}_2 \rangle; \tag{53b}$$

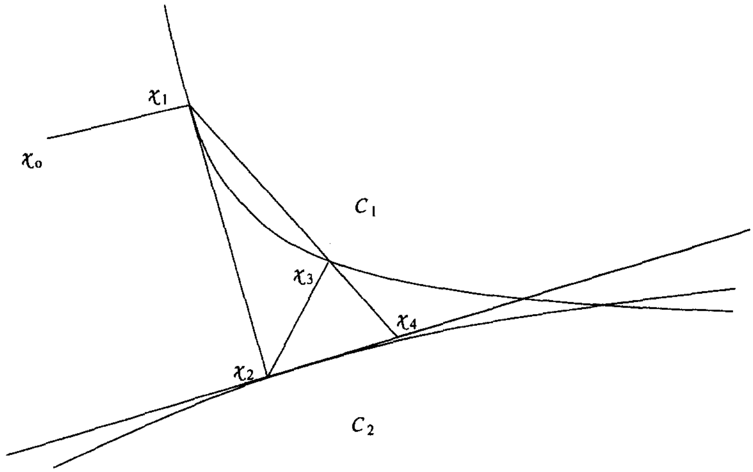

Fig. 2. Directional alternating projection algorithm.

$$\text{(2nd cycle)} \quad \mathscr{X}_5 = P_{\mathscr{C}_1}\mathscr{X}_4, \qquad \mathscr{X}_6 = P_{\mathscr{C}_2}\mathscr{X}_5, \qquad \mathscr{X}_7 = P_{\mathscr{C}_1}\mathscr{X}_6, \tag{53c}$$

$$\mathscr{X}_8 = \mathscr{X}_5 + \lambda_2(\mathscr{X}_7 - \mathscr{X}_5),$$

$$\lambda_2 = \|\mathscr{X}_5 - \mathscr{X}_6\|^2 / \langle \mathscr{X}_5 - \mathscr{X}_7, \mathscr{X}_5 - \mathscr{X}_6 \rangle; \tag{53d}$$

and so on. A schematical representation of the algorithm is given in Fig. 2. Note that, in each cycle, there are two projections on the same set ($\mathscr{C}_1$); the two projection points are used to derive a direction for the next point.

Convergence of the iterative scheme (53) to a point in the intersection $\mathscr{C}$ is guaranteed from the following result (Ref. 18).

Theorem 4.4. The sequence $\{\mathscr{X}_k\}_{k=0}^{\infty}$ provided by (53) converges to a vector $\mathscr{X}$ in $\mathscr{C} \triangleq \mathscr{C}_1 \cap \mathscr{C}_2 \cap \cdots \cap \mathscr{C}_n$ for any initial vector $\mathscr{X}_0 \in \mathscr{H}$.

Although there are no theoretical arguments about the rate of convergence of the directional alternating projection method, simple intuitive arguments based on Fig. 2 suggest that Algorithm (A3) will converge faster to a feasible solution than Algorithm (A1). Our numerical experience confirms this speculation. It can be verified easily that, when $\mathscr{C}_1$ and $\mathscr{C}_2$ are affine manifolds of the same dimension, the algorithm converges to the solution in one cycle, independently of the angle between them.

Another technique to possibly accelerate the rate of convergence of Algorithm (A1) is to use overrelaxation to extrapolate the projections $P_{\mathscr{C}_i}$ beyond the contours of the sets $\mathscr{C}_i$ (Ref. 18).

The feasibility problem (Q1) and the optimization problem (Q2) require the assumption that the intersection $\mathscr{C}_1 \cap \mathscr{C}_2 \cap \cdots \cap \mathscr{C}_n$ is nonempty. But often in practical applications, it is not known a priori that the sets $\mathscr{C}_i$ satisfy this assumption. It may well be the case that these sets correspond to inconsistent constraints; consequently, the nonempty intersection assumption might fail to hold. Hence, it is important that the iterative schemes proposed to solve these problems can recognize this consistency property. An answer to this question can be provided by the following result (Ref. 17), which is a generalization of Theorem 4.2.

Theorem 4.5. Suppose that at least one of the sets $\mathscr{C}_i$, $i=1, \ldots, n$, is bounded, and consider the sequence $\{\mathscr{X}_k\}_{k=0}^{\infty}$ defined in (A1). Then, the subsequence $\{\mathscr{X}_{nk}\}_{k=0}^{\infty}$ converges to a fixed point of the composition of the projections operator $\Phi \triangleq P_{\mathscr{C}_n} P_{\mathscr{C}_{n-1}} \ldots P_{\mathscr{C}_1}$.

Theorem 4.5 implies that each subsequence $\{\mathscr{X}_{nk+m}\}_{k=0}^{\infty}$, $m = 0, 1, \ldots, n-1$ [i.e., the subsequence which corresponds to the mth projection for every cycle in (A1)], converges to a point $\mathscr{X}_m^* \in \mathscr{C}_m$.

Note that Theorem 4.5 is valid even when the intersection $\mathscr{C}_1 \cap \mathscr{C}_2 \cap \cdots \cap \mathscr{C}_n$ is empty. Hence, combining this result with Theorem 4.2, we see that each of these subsequences converges to a common point $\mathscr{X}^*$ if and only if the sets have nonempty intersection and, in that case, $\mathscr{X}^*$ belongs to the intersection. Therefore, it can be tested numerically whether the iterative scheme (A1) converges to a solution of the feasibility problem (Q1), or if the corresponding sets have an empty intersection.

In the case where the constraints are inconsistent, we look to solve the infeasible optimization problem (Q3). For our purposes, the case of two disjoint sets is enough, although the results can be extended to more general situations. The next result shows that, for the case of two disjoint sets $\mathscr{C}_1$ and $\mathscr{C}_2$, the alternating projection method can provide the solution of the infeasible optimization problem (Q3).

Theorem 4.6. Consider the sequence $\{\mathscr{X}_k\}_{k=0}^{\infty}$ defined in (50) for $n = 2$, and suppose that at least one of the sets $\mathscr{C}_i$, $i=1, 2$, is bounded. Then, the subsequence $\{\mathscr{X}_{2k}\}_{k=0}^{\infty}$ converges to a point $\mathscr{X}^* \in \mathscr{C}_2$ which solves the infeasible optimization problem (Q3).

Therefore, the iterative scheme (50) provides a solution to the infeasible optimization problem (Q3), when $\mathscr{C}_1 \cap \mathscr{C}_2 = \varnothing$.

5. Projections Required for Covariance Design

In this section, the alternating convex projection methods described in Section 4 is utilized to solve the covariance design problems (37), (38), (42), (43). Consider the vector space $\mathcal{S}_{n_x}$ of $n_x \times n_x$ real symmetric matrices equipped with the inner product

$$\langle X, Y \rangle = \operatorname{tr}(XY), \qquad X, Y \in \mathcal{S}_{n_x}. \tag{54}$$

We derive analytical expressions for the projection operators on each one of the assignability and performance constrained sets $\mathcal{A}$, $\mathcal{P}_\epsilon$, $\mathcal{O}_i$, $\mathcal{C}$ defined in Section 3. To begin, we need the following definitions.

Definition 5.1. Define the symmetric vec operator $\operatorname{vec}_s: \mathcal{S}_{n_x} \rightarrow \mathbb{R}^{n_x(n_x+1)/2}$ as follows for any $X = (X_{ij}) \in \mathcal{S}_{n_x}$:

$$\begin{aligned} \operatorname{vec}_s X = [&X_{11}, \sqrt{2}X_{12}, \sqrt{2}X_{13}, \ldots, \sqrt{2}X_{1n_x}, X_{22}, \sqrt{2}X_{23}, \\ &\sqrt{2}X_{24}, \ldots, \sqrt{2}X_{2n_x}, \ldots, X_{n_x n_x}]^T. \end{aligned} \tag{55}$$

Hence, $\operatorname{vec}_s X$ contains the $n_x(n_x+1)/2$ distinct elements of the symmetric matrix X, where the nondiagonal elements are multiplied by a factor of $\sqrt{2}$. We will make use of vec_s to reduce the size of the projection problem from $n_x \times n_x$ to $n_x(n_x+1)/2$.

Proposition 5.1. The operators vec and vec_s are isometric isomorphisms of $\mathcal{S}_{n_x}$ onto $\mathbb{R}^{n_x(n_x+1)/2}$ and $\mathbb{R}^{n_x^2}$, respectively, where $\mathbb{R}^{n_x^2}$ and $\mathbb{R}^{n_x(n_x+1)/2}$ are assumed to be equipped with the standard Euclidean inner product $\langle \mathcal{X}, \mathcal{Y} \rangle = \Sigma \mathcal{X}_i \mathcal{Y}_i$.

Hence, vec and vec_s preserve the values of norms and inner products; i.e., for any $X, Y \in \mathcal{S}_{n_x}$,

$$\|X\| = \|\operatorname{vec} X\| = \|\operatorname{vec}_s X\|,$$

$$\langle X, Y \rangle = \langle \operatorname{vec} X, \operatorname{vec} Y \rangle = \langle \operatorname{vec}_s X, \operatorname{vec}_s Y \rangle,$$

where each norm and inner product should be interpreted in the appropriate space.

The following result, which can be verified easily, provides the connection between $\operatorname{vec}_s X$ and $\operatorname{vec} X$.

Proposition 5.2. Given any $X \in \mathcal{S}_{n_x}$, there exist a matrix $\Delta \in \mathbb{R}^{n_x^2 \times n_x(n_x+1)/2}$ such that

$$\operatorname{vec} X = \Delta \operatorname{vec}_s X. \tag{56}$$

Moreover, the matrix Δ is column orthogonal, i.e., $\Delta^T \Delta = I$.

Note that the columns of Δ are the column expansions of the standard orthonormal basis for the space $\mathcal{S}_{n_x}$. For example, for $n_x = 2$,

$$\Delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{57}$$

Now, we are ready to provide the expression for the projection operator on the affine manifold $\mathcal{A}$.

Proposition 5.3. Let $X_p \in \mathcal{S}_{n_x}$. The projection $X_p^* = P_{\mathcal{A}}(X_p)$ of X_p on the set $\mathcal{A}$, defined in (19), is given by the following expression using the vec operator:

$$\operatorname{vec} X_p^* = -\Delta(K\Delta)^+ \operatorname{vec}(E_p Q_p E_p) + \operatorname{vec} X_p, \tag{58}$$

where

$$E_p = I - B_p B_p^+, \qquad K = E_p \otimes E_p - (E_p A_p) \otimes (E_p A_p),$$

$$Q_p = X_p - A_p X_p A_p^T - D_p W_p D_p^T.$$

Proof. Since $\mathcal{A}$ is an affine subspace in $\mathcal{S}_{n_x}$, we need to show that $X_p^* - X_p$ is orthogonal to $X_p^* - \hat{X}_p$; i.e., that

$$\langle X_p^* - X_p, X_p^* - \hat{X}_p \rangle = 0, \tag{59}$$

for any $\hat{X}_p \in \mathcal{S}_{n_x}$. The inner product in (59) is equivalent to

$$\langle \Delta(K\Delta)^+ \operatorname{vec}(E_p Q_p E_p), \operatorname{vec} \hat{X}_p + \Delta(K\Delta)^+ \operatorname{vec}(E_p Q_p E_p) - \operatorname{vec} X_p \rangle, \tag{60}$$

where the expression (58) has been used for $\operatorname{vec} X_p^*$. Simple Kronecker product algebra manipulations reveal that

$$\operatorname{vec}(E_p Q_p E_p) = \operatorname{vec}(E_p D_p W_p D_p^T E_p) + K \operatorname{vec} X_p. \tag{61}$$

Hence, substituting in (60), we obtain

$$
\begin{aligned}
&\langle X_p^* - X_p, X_p^* - \hat{X}_p \rangle \\
&= \{\mathrm{vec}(E_p D_p W_p D_p^T E_p)\}^T (K\Delta)^{+T} \Delta^T \,\mathrm{vec}\, \hat{X}_p \\
&+ \{\mathrm{vec}\, X_p\}^T K^T (K\Delta)^{+T} \Delta^T \,\mathrm{vec}\, \hat{X}_p \\
&+ \{\mathrm{vec}(E_p D_p W_p D_p^T E_p)\}^T (K\Delta)^{+T} \Delta^T \Delta (K\Delta)^+ \,\mathrm{vec}(E_p D_p W_p D_p^T E_p) \\
&+ \{\mathrm{vec}(E_p D_p W_p D_p^T E_p)\}^T (K\Delta)^{+T} \Delta^T \Delta (K\Delta)^+ K \,\mathrm{vec}\, X_p \\
&+ \{\mathrm{vec}\, X\}^T K^T (K\Delta)^{+T} \Delta^T \Delta (K\Delta)^+ \,\mathrm{vec}(E_p D_p W_p D_p^T E_p) \\
&+ \{\mathrm{vec}\, X\}^T K^T (K\Delta)^{+T} \Delta^T \Delta (K\Delta)^+ K \,\mathrm{vec}\, X_p \\
&- \{\mathrm{vec}(E_p D_p W_p D_p^T E_p)\}^T (K\Delta)^{+T} \Delta^T \,\mathrm{vec}\, X_p \\
&- \{\mathrm{vec}\, X\}^T K^T (K\Delta)^{+T} \Delta^T \,\mathrm{vec}\, X_p. \qquad (62)
\end{aligned}
$$

Since $\hat{X}_p \in \mathscr{A}$, it obeys (6a) or (16a); hence, using Kronecker product algebra, we obtain the following expression for $\mathrm{vec}\, \hat{X}_p$:

$$\mathrm{vec}(E_p D_p W_p D_p^T E_p) = -K \,\mathrm{vec}\, \hat{X}_p.$$

Substituting this expression and the relations

$$\mathrm{vec}\, X_p = \Delta \,\mathrm{vec}_s\, X_p,$$
$$\mathrm{vec}\, \hat{X}_p = \Delta \,\mathrm{vec}_s\, \hat{X}_p$$

in (62), and using the fact that $\Delta^T \Delta = I$ and the defining properties of the Moore–Penrose generalized inverse, we finally obtain (59) after cancellations. □

The expression for the projection of any $X_p \in \mathscr{S}_{n_x}$ onto the set $\mathscr{P}_1$ defined in (20a) can be derived using a result from Ref. 28.

Proposition 5.4. Let $X_p \in \mathscr{S}_{n_x}$, and let $X_p - P = ULU^T$ be the eigenvalue–eigenvector decomposition of $X_p - P$, where P is defined by (21a) for the state feedback case and by (21b) for the dynamic controller case. The projection $X_p^* = P_{\mathscr{P}_1}(X_p)$ of X_p on the set $\mathscr{P}_1$, defined in (20a), is given by

$$X_p^* = UL_+ U^T + P, \qquad (63)$$

where L_+ is the diagonal matrix obtained by replacing the negative eigenvalues of $X_p - P$ in L by zero.

Hence, this projection requires an eigenvalue–eigenvector decomposition of the $n_x \times n_x$ symmetric matrix $X_p - P$. A similar expression holds for the projection $P_{\mathscr{P}_{2,\epsilon}}$ onto the set $\mathscr{P}_{2,\epsilon}$, by replacing P with ϵI in the above result.

The following result provides the expression for the projection operator on the set of the ith block output variance constraint $\mathscr{O}_i$.

Proposition 5.5. Let $X_p \in \mathscr{S}_{n_x}$. Consider the singular-value decomposition

$$C_{pi} = U_{pi}[\Sigma_{pi}, 0] V_{pi}^T, \tag{64}$$

and define

$$\bar{X}_{pi} \triangleq V_{pi}^T X_p V_{pi} = \begin{bmatrix} \bar{X}_{pi11} & \bar{X}_{pi12} \\ \bar{X}_{pi12}^T & \bar{X}_{pi22} \end{bmatrix}, \qquad \bar{X}_{pi11} \in \mathbb{R}^{n_{yi} \times n_{yi}}. \tag{65}$$

Consider the eigenvalue–eigenvector decomposition

$$\bar{X}_{pi11} - \Sigma_{pi}^{-1} U_{pi}^T (Y_i - H_{pi} W_p H_{pi}^T) U_{pi} \Sigma_{pi}^{-1} = W_i \Lambda_i W_i^T, \tag{66}$$

where Λ_i is a diagonal matrix which contains the eigenvalues of the matrix

$$\bar{X}_{pi11} - \Sigma_{pi}^{-1} U_{pi}^T (Y_i - H_{pi} W_p H_{pi}^T) U_{pi} \Sigma_{pi}^{-1}.$$

The projection $X_{pi}^* = P_{\mathscr{O}_i}(X_{pi})$ of X_p onto the set $\mathscr{O}_i$, defined in (28), is given by

$$X_{pi}^* = V_{pi} \begin{bmatrix} \bar{X}_{pi11}^* & \bar{X}_{pi12} \\ \bar{X}_{pi12}^T & \bar{X}_{pi22} \end{bmatrix} V_{pi}^T, \tag{67}$$

where

$$\bar{X}_{pi11}^* = W_i \Lambda_i^- W_i^T + \Sigma_{pi}^{-1} U_{pi}^T (Y_i - H_{pi} W_p H_{pi}^T) U_{pi} \Sigma_{pi}^{-1}, \tag{68}$$

and Λ_i^- is the diagonal matrix obtained by replacing the positive eigenvalues in Λ_i by 0.

Proof. Let

$$\hat{X}_{pi} \triangleq \begin{bmatrix} \hat{X}_{pi11} & \hat{X}_{pi12} \\ \hat{X}_{pi12}^T & \hat{X}_{pi22} \end{bmatrix} \in \mathscr{O}_i, \qquad \hat{X}_{pi11} \in \mathbb{R}^{n_{yi} \times n_{yi}}.$$

Consider the inner product defined in (49),

$$\langle X_{pi}^* - X_p, X_{pi}^* - \hat{X}_{pi} \rangle. \tag{69}$$

Since V_{pi} is an orthogonal matrix, (69) is equal to

$$\langle V_{pi}^T X_{pi}^* V_{pi} - V_{pi}^T X_p V_{pi},\ V_{pi}^T X_{pi}^* V_{pi} - V_{pi}^T \hat{X}_{pi} V_{pi} \rangle. \tag{70}$$

Define

$$V_{pi}^T \hat{X}_{pi} V_{pi} = \begin{bmatrix} \tilde{X}_{pi11} & \tilde{X}_{pi12} \\ \tilde{X}_{pi12}^T & \tilde{X}_{pi22} \end{bmatrix},$$

and note that (67) implies that

$$V_{pi}^T X_{pi}^* V_{pi} = \begin{bmatrix} \bar{X}_{pi11}^* & \bar{X}_{pi12} \\ \bar{X}_{pi12}^T & \bar{X}_{pi22} \end{bmatrix},$$

Hence, (70) is equal to

$$\left\langle \begin{bmatrix} \bar{X}_{pi11}^* & \bar{X}_{pi12} \\ \bar{X}_{pi12}^T & \bar{X}_{pi22} \end{bmatrix} - \begin{bmatrix} \bar{X}_{pi11} & \bar{X}_{pi12} \\ \bar{X}_{pi12}^T & \bar{X}_{pi22} \end{bmatrix}, \quad \begin{bmatrix} \bar{X}_{pi11}^* & \bar{X}_{pi12} \\ \bar{X}_{pi12}^T & \bar{X}_{pi22} \end{bmatrix} - \begin{bmatrix} \tilde{X}_{pi11} & \tilde{X}_{pi12} \\ \tilde{X}_{pi12}^T & \tilde{X}_{pi22} \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} \bar{X}_{pi11}^* - \bar{X}_{pi11} & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \bar{X}_{pi11}^* - \tilde{X}_{pi11} & \bar{X}_{pi12} - \tilde{X}_{pi12} \\ \bar{X}_{pi12}^T - \tilde{X}_{pi12}^T & \bar{X}_{pi22} - \tilde{X}_{pi22} \end{bmatrix} \right\rangle$$

$$= \langle \bar{X}_{pi11}^* - \bar{X}_{pi11}, \bar{X}_{pi11}^* - \tilde{X}_{pi11} \rangle. \tag{71}$$

Since $\hat{X}_{pi} \in \mathcal{O}_i$, we have

$$C_{pi} \hat{X}_{pi} C_{pi}^T \leq Y_i - H_{pi} W_p H_{pi}^T; \tag{72}$$

by substituting the singular-value decomposition (64) in (72), and after premultiplying and postmultiplying by U_{pi}^T and Σ_{pi}^{-1}, we obtain

$$[I \quad 0] V_{pi}^T \hat{X}_{pi} V_{pi} \begin{bmatrix} I \\ 0 \end{bmatrix} \leq \Sigma_{pi}^{-1} U_{pi}^T (Y_i - H_{pi} W_p H_{pi}^T) U_{pi} \Sigma_{pi}^{-1},$$

or

$$\tilde{X}_{pi11} \leq \Sigma_{pi}^{-1} U_{pi}^T (Y_i - H_{pi} W_p H_{pi}^T) U_{pi} \Sigma_{pi}^{-1}.$$

Hence, $\tilde{X}_{pi11}$ is an element of the set

$$\{\bar{X}_{pi11} \in \mathcal{S}_{n_{yi}} : \bar{X}_{pi11} \leq \Sigma_{pi}^{-1} U_{pi}^T (Y_i - H_{pi} W_p H_{pi}^T) U_{pi} \Sigma_{pi}^{-1}\}. \tag{73}$$

According to Proposition 5.2, the orthogonal projection of a matrix $\bar{X}_{pi11} \in \mathcal{S}_{n_{yi}}$ onto the set (73) is provided by the expression (68). Hence, the minimum distance condition (49) implies that the inner product in (71) is nonpositive. Hence,

$$\langle X_{pi}^* - X_p, X_{pi}^* - \hat{X}_{pi} \rangle \leq 0, \qquad \text{for any } \hat{X}_{pi} \in \mathcal{O}_i,$$

and this completes the proof. □

Next, the projection on the output cost constrained set $\mathscr{C}$ defined in (35) will be examined.

Proposition 5.6. Let $X_p \in \mathcal{S}_{n_x}$ and $\gamma > 0$ given. Then, the projection $X_p^* = P_{\mathscr{C}}(X_p)$ of X_p on $\mathscr{C}$ is given by

$$X_p^* = (1/\|R\|^2)(\gamma^* - \operatorname{tr}(X_p R))R + X_p, \tag{74}$$

where $R = C_p^T Q C_p$ and

$$\gamma^* = \min\{\gamma, \operatorname{tr}(X_p R)\}. \tag{75}$$

Proof. First, we note that $X_p^* \in \mathscr{C}$, since

$$\operatorname{tr}(X_p^* R) = (1/\|R\|^2)[\gamma^* - \operatorname{tr}(X_p R)] \operatorname{tr}(R^2) + \operatorname{tr}(X_p R) = \gamma^* \leq \gamma.$$

The validity of condition (49) can be verified since, for any $\hat{X}_p \in \mathscr{C}$, we obtain

$$\begin{aligned}
&\langle X_p^* - X_p, X_p^* - \hat{X}_p \rangle \\
&= (1/\|R\|^2) \operatorname{tr}\{[\gamma^* - \operatorname{tr}(X_p R)]R[1/\|R\|^2)[\gamma^* - \operatorname{tr}(X_p R)]R + X_p - \hat{X}_p]\} \\
&= (1/\|R\|^2)[\gamma^* - \operatorname{tr}(X_p R)][\gamma^* - \operatorname{tr}(\hat{X}_p R)].
\end{aligned}$$

Following similar arguments as in the proof of Proposition 5.3, we can conclude that

$$\langle X_p^* - X_p, X_p^* - \hat{X}_p \rangle \leq 0,$$

and the proof is complete. □

The above expression for the projections can be used to solve the covariance feasibility problem (P1) or the covariance optimization problem (P2) using the alternating projection algorithms of Section 4. The solution to the infeasible covariance optimization problem (P4) can be obtained by considering the sets

$$\mathscr{C}_1 = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m \cap \mathscr{C}, \qquad \mathscr{C}_2 = \mathscr{A} \cap \mathscr{P}_\epsilon,$$

and applying the result of Theorem 4.6. According to this result, an assignable covariance $X_p \in \mathscr{C}_2$ which closely approximates the desired constraints described by $\mathscr{C}_1$ is obtained by alternating projections on the sets $\mathscr{C}_1$ and $\mathscr{C}_2$. Note that, since each one of the sets $\mathscr{C}_i$ is the intersection of two or more sets, the alternating projection algorithm (A2) must be applied to get the projection onto the set $\mathscr{C}_i$. To find a solution to the covariance suboptimization problem (P3), we need the expression for the projection on the set $\mathscr{B}$ defined in (41). An expression for this projection is provided in Ref. 29.

Proposition 5.7. Let $X_p \in \mathcal{S}_{n_x}$ and $\delta > 0$. Then, the projection $X_p^* = P_{\mathcal{B}}(X_p)$ of X_p on $\mathcal{B}$ is given by

$$X_p^* = X_p, \qquad \text{if } \|X_p - X_p^*\| \leq \delta, \tag{76a}$$

$$X_p^* = (\delta / \|X_p - X_p^*\|)(X_p - X_p^*) + X_p^*, \qquad \text{if } \|X_p - X_p^*\| > \delta. \tag{76b}$$

When a desired plant-state covariance X_p is found, then Theorem 2.2 provides a parametrization of all static state feedback controllers, and Theorem 2.4 provides a parametrization of all dynamic output feedback controllers which assign this covariance to the closed-loop system. The free parameters in these parametrizations can be chosen such that the designed controller satisfies other objectives, such as minimum control effort; see Ref. 30.

6. Comments on the Reduced-Order Dynamic Controller Case

Although Theorem 2.4 parametrizes all dynamic controllers which assign a particular assignable plant covariance, it provides no systematic way to construct a reduced-order controller. The order of the dynamic controller n_c is equal to

$$n_c = \operatorname{rank}(X_p - \bar{X}_p),$$

where $\bar{X}_p$ is the positive-definite solution of the discrete Riccati equation (18b). Note that $\bar{X}_p$ depends on the choice of the free parameter $\bar{L} \in \mathbb{R}^{n_x \times n_z}$.

To design a fixed-order dynamic controller, we suggest the following procedure:

(i) Find a positive-definite matrix $\bar{X}_p$ by solving the Riccati equation (18b) for some free parameter $\bar{L} \in \mathbb{R}^{n_x \times n_z}$; note that the choice $\bar{L} = 0$ gives $\bar{X}_p = P$, where P is the positive-definite solution of the Riccati equation (17).

(ii) Compute an assignable plant covariance X_p which satisfies the desired performance objectives and in addition

$$X_p \geq \bar{X}_p, \tag{77}$$

$$\operatorname{rank}(X_p - \bar{X}_p) \leq n_c, \tag{78}$$

where n_c is the desired order of the reduced-order dynamic controller.

(iii) Using the results of Theorem 2.4, obtain a dynamic controller of order n_c which assigns X_p to the closed-loop system. To construct such a controller, compute the closed-loop covariance submatrices $X_c \in \mathbb{R}^{n_c \times n_c}$ and $X_{pc} \in \mathbb{R}^{n_x \times n_c}$ such that

$$X_{pc} X_c^{-1} X_{pc}^T = X_p - \bar{X}_p, \tag{79}$$

and form the closed-loop covariance

$$X = \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^T & X_c \end{bmatrix}. \tag{80}$$

A reduced-order dynamic controller of order n_c to satisfy the desired performance constraints is provided by (18c) for any choice of the unitary freedom U and the free parameter Z.

Our objective in this section is to provide an alternating projection technique to satisfy the constraints (77) and (78) in addition to the assignability and performance constraints discussed in the previous section. The constraint (77) for the plant covariance matrix X_p can be included in the plant covariance design problem by defining the set

$$\mathscr{P}_1 \triangleq \{X_p \in \mathscr{S}_{n_x} / X_p \geq \bar{X}_p\} \tag{81}$$

to replace the corresponding constraint set defined by (20a). The projection onto the set (81) is provided by (63), where P is replaced by $\bar{X}_p$. To satisfy the rank condition (78), we define the following subset of $\mathscr{S}_{n_x}$;

$$\mathscr{R}_{n_c} \triangleq \{X_p \in \mathscr{S}_{n_x}: \operatorname{rank}(X_p - \bar{X}_p) \leq n_c\}. \tag{82}$$

Hence, a solution to the n_cth order dynamic controller design problem, subject to output performance objectives, is guaranteed if a plant covariance X_p is obtained such that

$$X_p^* \in \mathscr{A} \cap \mathscr{P}_\epsilon \cap \mathscr{O} \cap \mathscr{C} \cap \mathscr{R}_{n_c}. \tag{83}$$

Following this formulation, covariance feasibility, optimization, and infeasible optimization problems for low-order control design x can be defined including the additional constraint $X_p \in \mathscr{R}_{n_c}$ in the covariance design problems (P1), (P2), (P3), (P4). However, $\mathscr{R}_{n_c}$ is not a convex set; this is easy to verify, since, for example, the $\mathbb{R}^{2 \times 2}$ matrices

$$J_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad J_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

have rank 1, but $(1/2)J_1 + (1/2)J_2$ has rank 2. Therefore, the alternating convex projection techniques of Section 5 cannot be used to guarantee a solution to the reduced-order controller design problem. In the following, we will extend the ideas of the alternating projection methods to include nonconvex constraint sets. Although these generalizations cannot guarantee global convergence, as in the convex case, our numerical experience indicates that these methods can perform well in practice.

To begin, we generalize the concept of a projection defined in Section 4, to the case of nonconvex sets. Consider a Hilbert space $\mathscr{H}$, a set $\mathscr{K}$ in $\mathscr{H}$, and a given vector $\mathscr{X}_0$. We call a vector $\mathscr{X}^*$ in $\mathscr{K}$ the projection of $\mathscr{X}_0$ onto $\mathscr{K}$ if it solves the following minimization problem

$$\|\mathscr{X}^* - \mathscr{X}_0\| \leq \|\hat{\mathscr{X}} - \mathscr{X}_0\|, \qquad \text{for any } \hat{\mathscr{X}} \in \mathscr{K}. \tag{84}$$

We still denote this projection by $\mathscr{X}^* = P_{\mathscr{K}}(\mathscr{X}_0)$. Note that the minimum is always attained when $\mathscr{K}$ is closed, but it might not be unique, since $\mathscr{K}$ is not convex. Hence, the corresponding projection operator $P_{\mathscr{K}}: \mathscr{H} \to \Pi(\mathscr{K})$ is a set-valued map onto a subset of $\mathscr{K}$, where $\Pi(\mathscr{K})$ denotes the family of all subsets of $\mathscr{K}$. For example, in the case where $\mathscr{H} = \mathscr{S}_2$, the matrices J_1 and J_2 defined above are both projections of the identity matrix on the set of symmetric matrices of rank one.

Now, suppose that we seek to solve the feasibility problem (Q1) for the case where some sets $\mathscr{C}_i$ are not convex. It can be shown that the standard alternating projection method given by Algorithm (A1) converges to a point in the intersection when the starting point $\mathscr{X}_0$ is in the vicinity of the intersection; see Ref. 31. Note that, for the projection onto a nonconvex set, any point of the projection map defined by (84) can be used.

The following result provides the expression for the projection map onto the fixed rank set $\mathscr{R}_{n_c}$ (Ref. 27).

Proposition 6.1. Let X_p be a given matrix in $\mathscr{S}_{n_x}$, and let $X_p - \bar{X}_p = U_1 \Sigma U_2^T$ be the singular value decomposition of $X_p - \bar{X}_p$. The projection $X_p^* = P_{\mathscr{R}_k}(X_p)$ of X_p on the set $\mathscr{R}_k$ is given by

$$X_p^* = U_1 \Sigma_k U_2^T + \bar{X}_p, \tag{85}$$

where Σ_k is the diagonal matrix obtained by replacing the $n_x - k$ smallest singular values of Σ by zero.

This projection can be combined easily with the projection on the set $\mathscr{P}_1$ defined in (81). In this case, in the eigenvalue–eigenvector decomposition of $X_p - \bar{X}_p$, the negative eigenvalues should be replaced by zero; in addition, if there are less than k zero eigenvalues, then the smallest nonzero ones should be replaced by zero to have a total sum of k zero eigenvalues.

The suggested approach to find a fixed-order dynamic controller is to solve first the covariance control design problem that corresponds to the full-order dynamic controller case $n_c = n_x$. Then, use this plant covariance matrix as a starting point to obtain a plant covariance matrix that corresponds to a controller of order $n_c = n_x - 1$, using the alternating projection techniques. Continue to design plant covariance matrices that correspond to reduced-order covariance controllers of order $n_c = n_x - 2, n_c = n_x - 3, \ldots$ using the plant covariance matrix of the previous solution as a starting point. Note that, even if the algorithm does not converge for a specified controller order $n_c = k$, a solution might exist since convergence of the algorithm is not guaranteed. However, our experience with the algorithm indicates that this approach performs well in practice, since it provides a better initial point to start the alternating projection algorithm.

7. Numerical Example

Consider the state space model of a fighter aircraft provided in Ref. 32, discretized with sampling period $T = 0.001$ sec,

$$A_p = \begin{bmatrix} 1.0000 & -0.0367 & -0.0189 & -0.0321 & 0.0035 & -0.0100 \\ 0.0000 & 0.9981 & 0.0010 & -0.0000 & -0.0002 & 0.0000 \\ 0.0000 & 0.0117 & 0.9974 & 0.0000 & -0.0311 & 0.0220 \\ 0.0000 & 0.0000 & 0.0010 & 1.0000 & -0.0000 & 0.0000 \\ 0 & 0 & 0 & 0 & 0.9704 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.9704 \end{bmatrix},$$

$$B_p = D_p = \begin{bmatrix} 0.0001 & -0.0000 \\ -0.0000 & 0.0000 \\ -0.0005 & 0.0003 \\ -0.0000 & 0.0000 \\ 0.0296 & 0 \\ 0 & 0.0296 \end{bmatrix}, \qquad C_p = M_p = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$W_p = 0.1, \qquad V = 1.$$

We seek a reduced-order dynamic controller to satisfy the following output performance constraints

$$\lim_{t \to \infty} \mathscr{E} y_1^2(t) \le 0.0015, \qquad \lim_{t \to \infty} \mathscr{E} y_2^2(t) \le 0.0030;$$

i.e., the plant covariance constraint set $\mathscr{O}$ has the form

$$\mathscr{O} = \{X_p \in \mathscr{S}_6 : (X_p)_{22} \le 0.0015, (X_p)_{44} \le 0.0030\}.$$

We seek to solve the covariance feasibility problem (P1), i.e., to find an assignable covariance X_p such that $X_p \in \mathcal{O}$. In addition, by choosing $\bar{X}_p = P$ [i.e., choosing the matrix freedom $\bar{L} = 0$ in (18b)] and requiring that X_p is in the constraint set

$$\mathcal{R}_k \triangleq \{X_p \in \mathcal{S}_{n_x} : \operatorname{rank}(X_p - \bar{X}_p) = n_c\}, \qquad n_c < 6,$$

we will restrict the order of the dynamic controller to be less than the order of the plant. For $n_c = 4$, the directional alternating projection algorithm (A3) provides the following feasible plant covariance:

$$X_p = \begin{bmatrix} 11.9218 & -0.0092 & -0.1082 & -0.0844 & -0.1987 & 0.1147 \\ -0.0092 & 0.0015 & 0.0048 & -0.0016 & 0.0112 & -0.0035 \\ 0.1082 & 0.0048 & 0.4422 & -0.0000 & -0.0125 & 0.0320 \\ -0.0844 & -0.0016 & -0.0000 & 0.0028 & 0.0105 & -0.0057 \\ -0.1987 & 0.0112 & -0.0125 & 0.0105 & 1.0119 & -0.0106 \\ 0.1147 & -0.0035 & 0.0320 & -0.0057 & -0.0106 & 1.0097 \end{bmatrix}.$$

The method required 73 iterations for an error tolerance 10^{-6}. Note that the corresponding output covariance matrix is

$$Y = \begin{bmatrix} 0.0015 & 0.0016 \\ 0.0016 & 0.0028 \end{bmatrix};$$

hence, the desired output performance constraints are satisfied. Specifically, the output variance constraint which corresponds to the first output y_1 is binding (i.e., is reaching the allowed bound 0.0015), although the one which corresponds to y_2 is not binding. Also, it can be verified easily that $X_p \in \mathcal{R}_4$; i.e., $\operatorname{rank}(X_p - P) = 4$, where P is the positive-definite solution of the Riccati equation (21b). Hence, a dynamic controller of order 4 can be designed. The closed-loop covariance matrix X can be assembled by choosing the controller covariance $X_c = I_4$ and the cross-covariance submatrix X_{pc} to satisfy (18a). A 4th-order dynamic controller to assign X to the closed-loop system is provided by (18c), where the orthogonal free matrix U is chosen to be the identity matrix. The resulting controller is the following:

$$A_c = \begin{bmatrix} 0.9426 & -0.0080 & -0.0440 & -0.0185 \\ -0.0896 & -0.9318 & 0.0230 & -0.0039 \\ 0.0670 & 0.0057 & 0.9957 & 0.0069 \\ 0.0235 & -0.0011 & -0.0079 & 0.9988 \end{bmatrix},$$

$$B_c = \begin{bmatrix} -0.1281 & -0.2903 \\ -0.3102 & -0.1705 \\ 0.0399 & 0.0439 \\ -0.0163 & -0.0423 \end{bmatrix},$$

$$C_c = \begin{bmatrix} -0.7354 & -3.2151 \\ -0.4870 & -1.2543 \\ 1.7746 & 0.2779 \\ 0.4010 & -0.4467 \end{bmatrix}, \quad D_c = \begin{bmatrix} -2.1253 & -11.2655 \\ -4.9299 & -10.3690 \end{bmatrix}.$$

Note that the alternating projection algorithm did not converge for $n_c < 4$; we were not able to design a dynamic controller of order less than 4 to satisfy the desired performance objectives.

An application of the discrete-time covariance control design methodology to the Hubble Space Telescope pointing control system design problem can be found in Ref. 29.

8. Conclusions

Covariance control theory allows the control design problem to be formulated in terms of assigning a plant-state covariance matrix X_p to the closed-loop system. This paper shows that the design problem in terms of X_p has some desirable convex properties. The corresponding assignability and performance constrained sets are convex sets; moreover, expressions for the projections on these sets are derived analytically. This allows the use of alternating convex projection methods to solve the covariance design problem. Improved convergence rates are obtained by using directional information. The algorithms presented can solve the following problems by convex programming: a covariance feasibility problem (covariance assignability subject to output system performance constraints); a covariance optimization problem (covariance assignability subject to output performance constraints such that the covariance is as close as possible to a desired one); and an infeasible covariance optimization problem (the closest assignable covariance to a given performance constraint set is obtained). Also, the case of a reduced-order dynamic controller design is discussed, and an alternating projection approach is suggested to attack this problem. Although convergence of the algorithm is not guaranteed in this case, numerical experiments have shown a satisfactory performance. A 6th-order numerical example is used to illustrate the design procedure, where a 4th-order controller is designed to satisfy given output performance objectives.

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