9
Singularity theory and nonlinear bifurcation analysis

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In this chapter we provide an introductory exposition of singularity theory and its application to nonlinear bifurcation analysis in elasticity. Basic concepts and methods are discussed with simple mathematics. Several examples of bifurcation analysis in nonlinear elasticity are presented in order to demonstrate the solution procedures.

9.1 Introduction

Singularity theory is a useful mathematical tool for studying bifurcation solutions. By reducing a singular function to a simple normal form, the properties of multiple solutions of a bifurcation equation can be determined from a finite number of derivatives of the singular function. Some basic ideas of singularity theory were first conjectured by R. Thom, and were then formally developed and rigorously justified by J. Mather (1968, 1969a, b). The subject was extended further by V. I. Arnold (1976, 1981). In two volumes of monographs, M. Golubitsky and D. G. Schaeffer (1985), and M. Golubitsky, I. Stewart and D. G. Schaeffer (1988) systematized the development of singularity theory, and combined it with group theory in treating bifurcation problems with symmetry. Their work establishes singularity theory as a comprehensive mathematical theory for nonlinear bifurcation analysis.

The purpose of this chapter is to give a brief exposition of singularity theory for researchers in elasticity. The emphasis is on providing a working knowledge of the theory to the reader with minimal mathematical prerequisites. It can also serve as a handy reference source of basic techniques and useful formulae in bifurcation analysis. Throughout this chapter, important solution techniques are demonstrated through examples and case studies. In most cases, theorems are stated without proof. No attempt has been made to provide a complete bibliographical survey of the field, nor an accurate account of the contributions made by various researchers.

To illustrate some topics dealt with in singularity theory, we begin with a classical example of nonlinear bifurcation in elasticity – the problem of the *elastica*. Consider the deformation of a slender elastic rod, subjected to a pair of compressive forces at the ends. By the Bernoulli-Euler beam theory, the bending moment is proportional to the curvature. This leads to the governing equation

$$EIu''(s) + \lambda \sin u(s) = 0, \qquad 0 < s < l, \tag{1.1}$$

where u is the angle between the undeformed rod and the tangent of the deformed rod, s the material coordinate, E the elastic modulus, I the moment of inertia, λ the compressive applied force, and l the length of the rod. The rod is hinged at its two ends, with the boundary conditions

$$u'(0) = u'(l) = 0. \tag{1.2}$$

The solution of this boundary value problem can be represented by using elliptic integrals (see, e.g., Timoshenko and Gere 1961, Section 2.7). In an elementary analysis, one considers the linearized equation of (1.1), namely

$$EIu''(s) + \lambda u(s) = 0. \tag{1.3}$$

The linearized boundary value problem (1.3) and (1.2) has a non-trivial solution

$$u(s) = C \cos \frac{n\pi s}{l} \tag{1.4}$$

if and only if

$$\lambda = \frac{n^2\pi^2 EI}{l^2}, \tag{1.5}$$

where n is an integer. The smallest non-zero value $P_{cr} = \pi^2 EI/l^2$ is usually called the *critical load* at which the rod is thought to start to buckle.

As the solution (1.4) and (1.5) of the linearized equation provides certain important information about the exact solution of the nonlinear problem, there are questions which cannot be answered by the linearized analysis. Some of the issues are discussed below, demonstrated with simple examples.

(i) It is well-known that the existence of non-trivial solutions of the linearized equation is a necessary condition for bifurcation. It is, however, not sufficient. Consider the algebraic equation

$$x^3 + \lambda^2 x = 0 \tag{1.6}$$

with a real state variable x and a real bifurcation parameter λ. Equation

(1.6) has the trivial solution $x = 0$ for all values of λ. The linearized equation about the trivial solution is

$$\lambda^2 x = 0,$$

which, at $\lambda = 0$, has non-trivial solutions $x = C$ for arbitrary C. The original nonlinear equation, however, does not have a bifurcation solution branch at $\lambda = 0$.

(ii) When a bifurcation solution branch does exist, little can be said about its qualitative behavior on the basis of the solution of the linearized equation. Qualitative behavior includes, for example, the number of the solution branches, and how they evolve as the bifurcation parameter varies. (Do they disappear or continue as the parameter decreases or increases from the bifurcation point?) Consider, for instance, the equation

$$x^5 - 3\lambda x^3 + 2\lambda^2 x = 0. \tag{1.7}$$

It again has the trivial solution $x = 0$ for all values of λ. The linearized equation about the trivial solution is

$$2\lambda^2 x = 0,$$

which, at $\lambda = 0$, has one non-trivial solution branch $x = C, C$ being an arbitrary constant. No information can be deduced from this linearized equation in regard to how many solution branches the nonlinear equation (1.7) actually has, and to the behavior of these solution branches in a neighborhood of the bifurcation point. By solving (1.7) directly, one finds that it has only the trivial solution when $\lambda < 0$, and has four non-trivial solution branches when $\lambda > 0$.

(iii) A mathematical equation often represents an idealization of a real physical system, which may have imperfections that are not accounted for by the equation. As a result, the behavior of the real system may be different, sometimes drastically, from that predicted by the solution. In reality, there are likely to be infinitely many ways in which imperfections could be present in a physical system. For the example of the compressed rod, possible imperfections include eccentric loads, imperfect supports, imperfect geometry, non-uniform elastic modulus, etc. Nevertheless, mathematically it is possible to describe the qualitative behavior of the imperfect system by introducing only a finite number of parameters. This is the subject of universal unfolding theory. As an example, a universal unfolding of (1.6) is given by

$$x^3 + \lambda^2 x + \alpha x = 0. \tag{1.8}$$

The solution diagrams of equation (1.8) are shown in Figure 1 for various values of α. These diagrams give a complete description of the qualitative behavior of *all* physical systems that are obtained by introducing small perturbations to the idealized system which is described by equation (1.6).

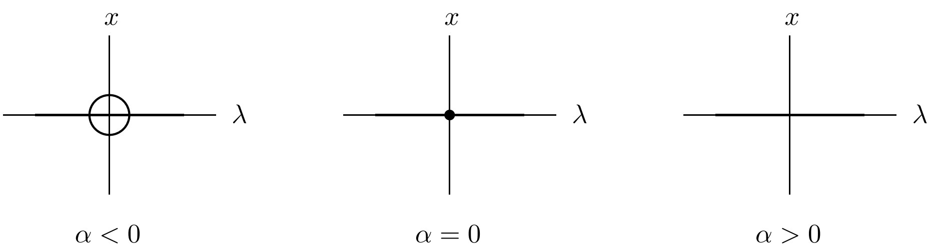

Figure 1. Bifurcation diagrams of equation (1.8).

Singularity theory is developed to address systematically the issues raised above. It provides efficient tools for studying nonlinear bifurcation problems. For example, by using the Liapunov-Schmidt reduction it can be shown that the solution set of equation (1.1), which is defined on an infinite-dimensional space, is equivalent to that of an algebraic equation with one state variable. By solving a recognition problem, it can be shown further that this algebraic equation is equivalent to a polynomial equation that has a pitchfork bifurcation. This implies that in a neighborhood of $(u(s), \lambda) = (0, n^2\pi^2 EI/l^2)$, the boundary value problem (1.1) and (1.2) has only the trivial solution when $\lambda < n^2\pi^2 EI/l^2$, and three solutions when $\lambda > n^2\pi^2 EI/l^2$ corresponding to the unbuckled state and two buckled states. Moreover, through an analysis of universal unfolding, it can be shown that two additional parameters are all that is needed to describe the qualitative behavior of the solutions in the presence of *all* possible imperfections.

Another topic dealt with in singularity theory is the bifurcation of systems with symmetry. The governing equations of a system with symmetry are equivariant under certain group actions. Singularity theory, when combined with group theory, provides a powerful tool in gaining insight into the persistence of or change of the symmetry that a solution branch possesses at the bifurcation point.

Singularity theory also concerns the stability of bifurcation solution branches. Often used is the so-called linear stability criterion. Under such a criterion, the reduced algebraic equation is treated as the governing equation for the equilibrium states of a dynamical system, and the stability of the solutions is determined by examining the behavior of this system under small dynamic disturbances.

In the next section, we introduce the basic terminologies and formulation of bifurcation problems. The derivation of the linearized equation is discussed. The procedure of the Liapunov-Schmidt reduction is then presented. Section 9.3 is devoted to the solution method of the recognition problem, which constitutes the main technical development of singularity theory. The notions of equivalence and normal form are introduced. The normal form of pitchfork bifurcation is then derived explicitly through elementary analysis, and related concepts are discussed in a general context. The solutions of some recognition problems are listed. Also given is a brief discussion of universal unfolding theory. Three case studies are presented in the remaining sections. In Section 9.4, bifurcation analysis of pure homogeneous deformations with Z_2 symmetry is presented. A similar problem with D_3 symmetry is studied in Section 9.5. This difference in symmetry, however, results in distinct representations of the singular function and bifurcation diagrams from those in Section 9.4. In the concluding Section 9.6, the solution of an infinite-dimensional bifurcation problem, that of the inflation of a spherical membrane, is discussed.

9.2 Bifurcation equation and Liapunov-Schmidt reduction

In this section, we formulate bifurcation problems and discuss a necessary condition for the existence of bifurcation solution branches by examining the linearized equation. A brief exposition is given of the Liapunov-Schmidt reduction scheme, which reduces the bifurcation equation to a finite-dimensional or lower dimensional algebraic equation.

9.2.1 Bifurcation equation. Linearized equation

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces (i.e. normed linear spaces), $\mathcal{U}$ an open subset of $\mathcal{X}$, and Λ an open subset of $\mathbb{R}^n$. Consider a smooth mapping $f : \mathcal{U} \times \Lambda \to \mathcal{Y}$. We assume that the equation

$$f(u, \lambda) = 0 \tag{2.1}$$

determines the state of a physical system with n parameters. For example, (2.1) can be the equilibrium equation for an elastic body. In this connection, u can be a function that describes the deformation of the body, f a differential operator, and λ a set of parameters that specify, for example, the loads, the geometry and the material properties of the body. The variable u is called the *state variable*, and λ the *bifurcation parameter*.

As an example, let us re-formulate the elastica problem discussed in Section 9.1. Let

$$\mathcal{U} \subset \mathcal{X} \equiv \{u \in C^2([0,l];\mathbb{R}) : u'(0) = u'(l) = 0\}, \tag{2.2}$$

and let $f : \mathcal{U} \times \mathbb{R} \to C^0([0,l];\mathbb{R})$ be given by

$$f(u(s),\lambda) = EIu''(s) + \lambda \sin u(s). \tag{2.3}$$

Then the boundary value problem (1.1) and (1.2) can be expressed by (2.1) with $u \in \mathcal{U}$. In this example, the elastic modulus E and the moment of inertia I are taken to be constant. Alternatively, they can be treated as additional bifurcation parameters, although here their effects are essentially inseparable from that of λ.

Suppose that $(u_0, \lambda_0) \in \mathcal{U} \times \Lambda$ satisfies (2.1). If the number of solutions of (2.1) in an arbitrarily small neighborhood of (u_0, λ_0) changes as λ varies, the pair (u_0, λ_0) is called a *bifurcation point*. The solutions in this neighborhood are referred to as *bifurcation solution branches*, and a graphical representation of the bifurcation solution branches is called a *bifurcation diagram*.

The mapping $f(u,\lambda)$ is assumed to be smooth in the sense that it has Fréchet derivatives of any order. The first-order Fréchet derivative $D_u f(u_0,\lambda_0)$ of $f(u,\lambda)$ with respect to u at (u_0,λ_0) is a linear operator from $\mathcal{X}$ to $\mathcal{Y}$ such that

$$f(u,\lambda_0) = f(u_0,\lambda_0) + D_u f(u_0,\lambda_0)(u-u_0) + o(\|u-u_0\|) \quad \text{as } \|u-u_0\| \to 0.$$

Higher-order Fréchet derivatives of $f(u,\lambda)$ are defined similarly. By the implicit function theorem, a necessary condition for (u_0,λ_0) to be a bifurcation point is that $D_u f(u_0,\lambda_0)$ be not invertible.

For the example of f given by (2.3), the pair $(0,\lambda_0)$ is a solution of (2.1) for each $\lambda_0 \in \mathbb{R}$. The Fréchet derivative of f at $(0,\lambda_0)$ is given by

$$D_u f(0,\lambda_0)u(s) = EIu''(s) + \lambda_0 u(s).$$

The operator $D_u f(0,\lambda_0)$ is not invertible if and only if the boundary value problem (1.3) and (1.2) has a non-trivial solution. This occurs if and only if

$$\lambda_0 = \frac{n^2\pi^2 EI}{l^2},$$

as observed earlier.

9.2.2 Liapunov-Schmidt reduction

For many bifurcation problems in elasticity, the space $\mathcal{X}$ of the state variable is high dimensional or even infinite dimensional, as is the case with (2.2). This is one of the sources of difficulty in solving bifurcation problems. There is, however, a standard procedure, called the *Liapunov-Schmidt reduction*, that may effectively reduce an infinite-dimensional bifurcation problem to one with finite dimensions, or reduce a high-dimensional problem to one with lower dimensions.

The basic idea of the Liapunov-Schmidt reduction is to decompose (2.1) into two equivalent equations. One has finite dimensions, or lower dimensions if (2.1) is already finite dimensional. The other equation can be solved by using the implicit function theorem. Substitution of the solution of the second equation into the first equation results in a reduced equation that is equivalent to (2.1).

The Liapunov-Schmidt reduction is applicable when the Fréchet derivative of f at the bifurcation point is a Fredholm operator. Precisely, a bounded linear operator $L : \mathcal{X} \to \mathcal{Y}$ is *Fredholm* if the kernel of L, defined by ker $L \equiv \{u \in \mathcal{X} : L(u) = 0\}$, is finite-dimensional, and if the range of L, defined by range $L \equiv \{y \in \mathcal{Y} : L(u) = y \text{ for some } u \in \mathcal{X}\}$, is closed in $\mathcal{Y}$ with a finite-dimensional complement. The *index* $i(L)$ of a Fredholm operator is given by

$$i(L) = \dim \ker L - \operatorname{codim} \operatorname{range} L,$$

where codim denotes the dimension of the complement. Most operators encountered in elasticity are Fredholm of index 0, which we shall assume in this chapter. A consequence of this assumption is that the spaces $\mathcal{X}$ and $\mathcal{Y}$ have the orthogonal decompositions

$$\mathcal{X} = \ker L \oplus \mathcal{K}, \qquad \mathcal{Y} = \operatorname{range}\mathcal{L} \oplus \mathcal{R}, \tag{2.4}$$

with $\dim \mathcal{R} = \dim \ker L$.

We now consider equation (2.1). Suppose that $(0, \lambda_0) \in \mathcal{X} \times \Lambda$ is a solution of (2.1), so that

$$f(0, \lambda_0) = 0, \tag{2.5}$$

and that $L \equiv D_u f(0, \lambda_0)$ is a Fredholm operator of index 0 with $\dim \ker L > 0$. Let $P : \mathcal{Y} \to \operatorname{range} L$ be the orthogonal projection associated with the decomposition (2.4). We can write equation (2.1) as an equivalent pair of equations

$$Pf(v + w, \lambda) = 0, \tag{2.6}$$

$$(I - P)f(v + w, \lambda) = 0, \tag{2.7}$$

where I is the identity operator, and we have replaced u by $v+w$ with $v \in \ker L$

and $w \in \mathcal{K}$. Define $F : \ker L \times \mathcal{K} \times \Lambda \to \operatorname{range} L$ by

$$F(v, w, \lambda) \equiv Pf(v + w, \lambda).$$

It is observed that the linear operator $D_w F(0, 0, \lambda_0) : \mathcal{K} \to \operatorname{range} L$ is the restriction of L on $\mathcal{K}$. This linear operator is invertible. By the implicit function theorem, one can solve equation (2.6) locally for w. That is, there is a smooth function W defined in a neighborhood $\mathcal{N}$ of $(0, \lambda_0)$ in $\ker L \times \Lambda$, such that

$$Pf(v + W(v, \lambda), \lambda) = 0 \quad \forall (v, \lambda) \in \mathcal{N}. \tag{2.8}$$

Roughly speaking, by projecting the state variable and the value of f on $\mathcal{K}$ and range L, respectively, equation (2.6) effectively factors out the non-invertible part of f. If follows from (2.5) and (2.8) that

$$W(0, \lambda_0) = 0. \tag{2.9}$$

Now define $g : \mathcal{N} \to \mathcal{R}$ by

$$g(v, \lambda) = (I - P)f(v + W(v, \lambda), \lambda).$$

By the construction of W, equation

$$g(v, \lambda) = 0 \tag{2.10}$$

is then equivalent to (2.6) and (2.7), and therefore equivalent to equation (2.1). Equation (2.10) is called the *reduced bifurcation equation*. Its solutions are in one-to-one correspondence with those of (2.1). It is noted that the state variable v of the reduced equation is in the finite-dimensional space ker L.

In loose terms, the fact that equation (2.1) admits multiple solutions near a bifurcation point means that it is impossible to express all components of the state variable uniquely in terms of the bifurcation parameter. The advantage offered by (2.6) is that a unique solution of this equation is ensured by the implicit function theorem. This amounts to solving (2.1) for as many components of the state variable as possible in terms of the bifurcation parameter and the remaining components of the state variable. Substituting the resulting components back into (2.1) yields (2.10).

It is noted that equation (2.6), just like (2.1), may be a nonlinear equation and cannot be solved explicitly. As a result, the explicit form of g in (2.10) cannot be obtained in general. This information, however, is not at all necessary when one studies the qualitative behavior of the solution of (2.10) by using singularity theory. As shall be discussed in Section 9.3, what one needs are no more than the values of a few derivatives of g at the bifurcation point. These derivatives can be found by applying the chain rule and the implicit function theorem to (2.6) and (2.7).

To this end, let $\{e_i\}$ be an orthogonal basis of ker L, where $i \in \{1,..,n\}, n$ being the dimension of ker L. An element v of ker L can be written as

$$v = x_i e_i.$$

Here the usual summation convention for repeated indices is assumed. We can rewrite equation (2.8) as

$$Pf(x_i e_i + W(x_i,\lambda),\lambda) = 0. \tag{2.11}$$

Here we have used W for different functions. Differentiating (2.11) with respect to x_i and evaluating the resulting equation at $(v,\lambda) = (0,\lambda_0)$, we find that

$$PL(e_i + W_i) = 0, \quad \text{where } W_i \equiv \frac{\partial W}{\partial x_i}(0,\lambda_0). \tag{2.12}$$

Similar notation will be used in the sequel. Equation (2.12) implies that $L(e_i + W_i) = 0$. Hence, $(e_i + W_i) \in$ ker L. This further implies, since $W_i \in \mathcal{K}$, that

$$W_i = 0. \tag{2.13}$$

Next, differentiating (2.11) with respect to λ leads to

$$P(LW_\lambda + f_\lambda) = 0, \tag{2.14}$$

where f_λ is the Fréchet derivative of f with respect to λ, again evaluated at $(v,\lambda) = (0,\lambda_0)$. Equation (2.14) implies that

$$W_\lambda = -L^{-1}Pf_\lambda, \tag{2.15}$$

where L^{-1} is the inverse of the restriction of L on $\mathcal{K}$. A similar calculation by differentiating (2.11) with respect to x_i and x_j gives

$$W_{ij} = -L^{-1}Pf_{uu}e_i e_j. \tag{2.16}$$

Let $\{e_m^*\}$ be an orthogonal basis of $\mathcal{R}$ and define

$$g_m(x_i,\lambda) \equiv (e_m^*, f(x_i e_i + W(x_i,\lambda),\lambda)), \qquad m = 1,..,n, \tag{2.17}$$

where $(\cdot,\cdot)$ is the inner product on $\mathcal{Y}$. Equation (2.10) then can be written as

$$g_m(x_i,\lambda) = 0, \qquad m = 1,..,n.$$

It is obvious from (2.5), (2.9) and (2.17) that

$$g_m(0,\lambda_0) = 0, \tag{2.18}$$

which simply restates that $(0,\lambda_0)$ is a solution of (2.1). We now compute the derivatives of $g_m(x_i, \lambda)$. Differentiating (2.17) with respect to x_i and using the fact that e_m^* is orthogonal to range L, we find that

$$g_{m,i} = (e_m^*, L(e_i + W_i)) = 0. \tag{2.19}$$

Here and henceforth, unless otherwise stated, the derivatives of g_m are evaluated at $(v, \lambda) = (0, \lambda_0)$, and we use the notation

$$g_{m,i} \equiv \frac{\partial g_m}{\partial x_i}, \; g_{m,\lambda} \equiv \frac{\partial g_m}{\partial \lambda}, \text{ etc.} \tag{2.20}$$

Similar calculations, with the help of (2.13) and (2.15), lead to

$$g_{m,\lambda} = (e_m^*, LW_\lambda + f_\lambda) = (e_m^*, f_\lambda), \tag{2.21}$$

$$g_{m,ij} = (e_m^*, f_{uu}(e_i + W_i)(e_j + W_j) + LW_{ij}) = (e_m^*, f_{uu} e_i e_j), \tag{2.22}$$

$$\begin{aligned} g_{m,i\lambda} &= (e_m^*, f_{uu}(e_i + W_i)W_\lambda + f_{u\lambda}(e_i + W_i) + LW_{i\lambda}) \\ &= (e_m^*, -f_{uu} e_i (L^{-1} P f_\lambda) + f_{u\lambda} e_i), \end{aligned} \tag{2.23}$$

$$\begin{aligned} g_{m,ijk} &= (e_m^*, f_{uuu}(e_i + W_i)(e_j + W_j)(e_k + W_k) \\ &\quad + f_{uu}[(e_i + W_i)W_{jk} + (e_j + W_j)W_{ik} + (e_k + W_k)W_{ij}] + LW_{ijk}) \\ &= (e_m^*, f_{uuu} e_i e_j e_k + f_{uu}(e_i W_{jk} + e_j W_{ik} + e_k W_{ij})), \end{aligned} \tag{2.24}$$

where W_{ij} is given by (2.16).

As an example, let us examine equation (2.1) with $\mathcal{X}$ being given by (2.2), $\mathcal{Y} = C^0([0, l]; \mathbb{R}), \Lambda = \mathbb{R}$, and f given by (2.3). The Fréchet derivative L of f with respect to u at $(u(s), \lambda) = (0, \lambda_0)$ is given by

$$Lu(s) = EIu''(s) + \lambda_0 u(s).$$

By solving the two-point boundary value problem

$$Lu = 0, \qquad u \in \mathcal{X},$$

one finds that

$$\dim \ker L = \begin{cases} 1 & \text{if } \lambda_0 = n^2 \pi^2 EI/l^2 \\ 0 & \text{otherwise.} \end{cases}$$

We shall consider the case where $\lambda_0 = n^2 \pi^2 EI/l^2$. The subspace $\ker L$ is given by

$$\ker L = \{u \in \mathcal{X} : u(s) = C \cos \frac{n\pi s}{l}, C \in \mathbb{R}\}.$$

We shall employ the standard inner product

$$(u, v) = \int_0^l u(s)v(s)ds.$$

The orthogonal complement $\mathcal{K}$ of $\ker L$ in $\mathcal{X}$ is then given by

$$\mathcal{K} = \{w \in \mathcal{X} : \int_0^l w(s) \cos \frac{n\pi s}{l} ds = 0\}.$$

Moreover, an element $y(s)$ in the orthogonal complement $\mathcal{R}$ of range L in $\mathcal{Y}$ satisfies

$$\begin{aligned}(y, Lu) &= \int_0^l y(s)[EIu''(s) + \lambda_0 u(s)]ds \\ &= -[EIy'(s)u(s)]_0^l + \int_0^l [EIy''(s) + \lambda_0 y(s)]u(s)ds \\ &= 0 \quad \forall u \in \mathcal{X}\end{aligned}$$

Hence, $y(s)$ must satisfy

$$EIy''(s) + \lambda_0 y(s) = 0, \quad y'(0) = y'(l) = 0.$$

This result also follows readily from the fact that the linear operator L is self-adjoint. Therefore, for this particular example, we have

$$\mathcal{R} = \ker L, \ \text{range}\, L = \mathcal{K}.$$

Furthermore, the orthogonal projection of $\mathcal{Y}$ onto range L is given by

$$Py(s) = y(s) - [\frac{2}{l} \int_0^l y(t) \cos \frac{n\pi t}{l} dt] \cos \frac{n\pi s}{l}.$$

The subspaces $\ker L$ and $\mathcal{R}$ are one dimensional and spanned by the function

$$e = e^* = \cos \frac{n\pi s}{l}.$$

Hence, we shall use $g(x, \lambda)$ in place of $g_m(x_i, \lambda)$ in (2.17), etc. It follows from (2.3) that

$$f_\lambda = 0, \ f_{uu} = 0, \ f_{u\lambda} = 1, \ f_{uuu} = -\lambda_0. \tag{2.25}$$

Substituting $(2.25)_{1,2}$ into (2.15) and (2.16) gives

$$W_\lambda = 0, \ W_{xx} = 0.$$

By using the above results and (2.18), (2.19), (2.21)–(2.24), we find that

$$g = g_x = g_\lambda = g_{xx} = 0, \ g_{x\lambda} = (e^*, e) = \frac{l}{2}, \ g_{xxx} = (e^*, -\lambda_0 e^3) = -\frac{3l\lambda_0}{8}. \tag{2.26}$$

Other derivatives of $g(x, \lambda)$ can be calculated similarly. However, we shall see in the next section that the derivatives in (2.26) are sufficient to determine the qualitative behavior of the bifurcation solution branches in a neighborhood of the bifurcation point.

9.3 The recognition problem

The material presented in this section constitutes an essential part of singularity theory. A recognition problem for a given algebraic equation is to find a polynomial equation, as simple as possible, of which the solution is in one-to-one correspondence with that of the given equation in a neighborhood of the bifurcation point. This polynomial, called the *normal form* of the given function, can be determined solely by the values of a finite number of derivatives of the given function at the bifurcation point.

9.3.1 Equivalence and normal form

In this section, we consider the reduced bifurcation equation of a single state variable and a single bifurcation variable,

$$g(x, \lambda) = 0. \tag{3.1}$$

For convenience, we move the anticipated bifurcation point $(0, \lambda_0)$ to the origin by a translation of λ. Let $\mathcal{N}$ be a small neighborhood of the origin in $\mathbb{R} \times \mathbb{R}$. The function $g : \mathcal{N} \to \mathbb{R}$ is assumed to be of class C^∞ with

$$g(0, 0) = 0. \tag{3.2}$$

Henceforth, all the derivatives of g are evaluated at the origin. The analysis presented here is local. All conclusions are valid in $\mathcal{N}$.

A necessary condition for the existence of bifurcation of (3.1) at the origin is that

$$g_x(0, 0) = 0. \tag{3.3}$$

A function g that satisfies (3.2) and (3.3) is said to be singular at the origin. In this case, one cannot apply the implicit function theorem to solve (3.1) for x in terms of λ. The primary goal here is to find an equation, as simple as possible, whose solution has the same qualitative behavior as that of (3.1), and whose bifurcation diagram can easily be obtained by elementary computation. As commented earlier, the information that we have, and need, concerning g is no more than the values of its derivatives at the origin.

We first make precise the meaning of the phrase that the solutions of two equations have the same qualitative behavior. Two smooth functions $g, h : \mathcal{N} \to \mathbb{R}$ are said to be *strongly equivalent*† if there exist smooth functions $X, S : \mathcal{N} \to \mathbb{R}$, such that

$$X(0,0) = 0,\ X_x(x,\lambda) > 0,\ S(x,\lambda) > 0, \tag{3.4}$$

and that

$$g(x,\lambda) = S(x,\lambda) h(X(x,\lambda),\lambda) \qquad \forall (x,\lambda) \in \mathcal{N}. \tag{3.5}$$

A direct consequence of this definition is that there is a one-to-one correspondence between the solutions of (3.1) and

$$h(x,\lambda) = 0; \tag{3.6}$$

in particular, the number of solutions is preserved when (3.1) is replaced by (3.6). To see this, suppose that, for a fixed λ, equation (3.1) has exactly n solutions $x_1 < x_2 < ... < x_n$, i.e.

$$g(x,\lambda) = 0 \text{ iff } x = x_i, i = 1, .., n.$$

This and (3.5) imply that

$$h(X,\lambda) = 0 \text{ iff } X = X(x_i,\lambda), i = 1, .., n.$$

It is obvious from $(3.4)_2$ that $X(x_1,\lambda) < X(x_2,\lambda) < ... < X(x_n,\lambda)$.

As an example, we examine a function $g(x,\lambda)$ that satisfies (2.26). It will be shown in the next section that there are functions $X(x,\lambda)$ and $S(x,\lambda)$, satisfying (3.4), such that

$$g(x,\lambda) = S(x,\lambda)[\lambda X(x,\lambda) - X^3(x,\lambda)] \tag{3.7}$$

in a neighborhood of $(x,\lambda) = (0,0)$. Hence, function $g(x,\lambda)$ is equivalent to

$$h(x,\lambda) = \lambda x - x^3. \tag{3.8}$$

The solution of (3.1), and therefore the solution of the elastica problem, is in one-to-one correspondence with the solution of

$$\lambda x - x^3 = 0. \tag{3.9}$$

The bifurcation diagram of (3.9) corresponds to a pitchfork bifurcation as sketched in Figure 2. The function $h(x,\lambda)$ is called a *normal form* of $g(x,\lambda)$. Much of singularity theory is devoted to determining the simplest normal form for a function of which a certain number of derivatives are given at the bifurcation point. Some comments are in order here. Firstly, for an equation

† In the definition of general equivalence, the transformation of the bifurcation parameter λ is also allowed. See Golubitsky and Schaeffer (1985).

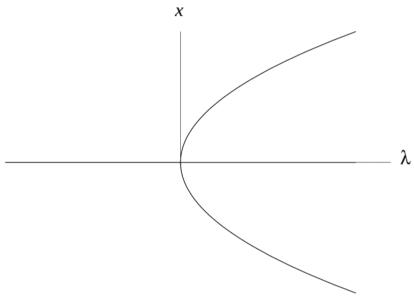

Fig. 2. Pitchfork bifurcation $\lambda x - x^3 = 0$.

as simple as (3.7), one can, by using the solution of cubic equations, actually find $X(x,\lambda)$ and $S(x,\lambda)$ for a given $g(x,\lambda)$ that satisfies (2.26). However, this elementary approach, aside from other matters such as the degree of sophistication, is not practical for more complicated problems. Secondly, it may appear plausible that the function $h(x,\lambda)$ could be obtained by inspecting (2.26) and retaining the non-zero leading terms of the Taylor expansion of $g(x,\lambda)$. A further examination of this idea quickly reveals that the situation is not as simple as it may appear at first glance. For example, it is not intuitive at all why $h(x,\lambda)$ need not contain the term λ^2 while $g_{\lambda\lambda}$ may not be zero. Moreover, other third order derivatives of g, i.e., $g_{xx\lambda}, g_{x\lambda\lambda}$ and $g_{\lambda\lambda\lambda}$, are not calculated and listed in (2.26). Why these derivatives can be safely omitted is entirely beyond what one can deduce from the usual Taylor expansion argument.

These are exactly the issues that singularity theory addresses, and resolves in a complete and elegant manner. Specifically, the conditions for the equivalence of two functions are established through a thorough examination of the tangent spaces of the two functions. It can be shown that function $h(x,\lambda)$ in (3.8) is in fact the simplest normal form of function $g(x,\lambda)$ that satisfies (2.26). Additional terms such as λ^2 are redundant. It can be shown further that all functions that are equivalent to $h(x,\lambda)$ of form (3.8) are characterized by the derivatives listed in (2.26). No additional derivatives are necessary.

9.3.2 Recognition problem for pitchfork bifurcation

The development of singularity theory draws heavily upon algebra theory. Some basic concepts would be best expounded with the theories of groups and rings. It is impractical to go into these theories in an introductory exposition. Here, we shall illustrate some pertinent ideas by solving the recognition problem for the pitchfork bifurcation using elementary mathematics. In particular, we shall show that a function $g(x,\lambda)$ is strongly equivalent to $h(x,\lambda)$ given by (3.8) if and only if

$$g = 0,\ g_x = 0,\ g_\lambda = 0,\ g_{xx} = 0,\ g_{x\lambda} > 0,\ g_{xxx} < 0. \tag{3.10}$$

Equations $(3.10)_{1-4}$ are called *defining conditions* for the normal form $\lambda x - x^3$, while inequalities $(3.10)_{5,6}$ are called *nondegeneracy conditions*.

The proof of necessity is elementary. Suppose that $g(x,\lambda)$ is strongly equivalent to $h(x,\lambda)$. By (3.5), there exist $X(x,\lambda)$ and $S(x,\lambda)$ satisfying (3.4), such that

$$g(x,\lambda) = S(x,\lambda)[\lambda X(x,\lambda) - X^3(x,\lambda)]. \tag{3.11}$$

Taking the derivatives of (3.11) successively and evaluating them at the origin with the help of (3.4) leads to (3.10).

We now turn to sufficiency. Let a smooth function $g(x,\lambda)$ satisfying (3.10) be given. By (3.10) and Taylor's theorem, $g(x,\lambda)$ can be written as

$$g(x,\lambda) = c_1\lambda x + c_2\lambda^2 + a_1(x,\lambda)x^3 + a_2(x,\lambda)\lambda x^2 + a_3(x,\lambda)\lambda^2 x + a_4(x,\lambda)\lambda^3, \tag{3.12}$$

where c_1 and c_2 are constants, and $a_1(x,\lambda),\ldots,a_4(x,\lambda)$ are smooth functions with

$$c_1 > 0,\ a_1(x,\lambda) < 0. \tag{3.13}$$

Here, we re-emphasize that the analysis is local, and all statements, such as $(3.13)_2$, are valid in the small neighborhood $\mathcal{N}$. We can rearrange the terms in (3.12) to give

$$g(x,\lambda) = a_5(x,\lambda)\lambda x + a_6(x,\lambda)\lambda^2 + a_1(x,\lambda)x^3, \tag{3.14}$$

where

$$a_5(x,\lambda) \equiv c_1 + a_2(x,\lambda)x + a_3(x,\lambda)\lambda > 0,\ a_6(x,\lambda) \equiv c_2 + a_4(x,\lambda)\lambda.$$

Assume that the right-hand side of (3.14) is strongly equivalent to

$$f(x,\lambda) \equiv a_5(x,\lambda)\lambda x + a_1(x,\lambda)x^3. \tag{3.15}$$

Then by choosing

$$X(x,\lambda) = \sqrt{-\frac{a_1(x,\lambda)}{a_5(x,\lambda)}}x,\ S(x,\lambda) = \sqrt{-\frac{a_5^3(x,\lambda)}{a_1(x,\lambda)}} \tag{3.16}$$

one has

$$f(x,\lambda) = S(x,\lambda)[\lambda X(x,\lambda) - X^3(x,\lambda)].$$

That is, $f(x,\lambda)$, and therefore $g(x,\lambda)$, is strongly equivalent to $h(x,\lambda)$. It is readily checked that functions $X(x,\lambda)$ and $S(x,\lambda)$ defined in (3.16) satisfy (3.4).

It remains to show that functions $g(x,\lambda)$ and $f(x,\lambda)$ in (3.14) and (3.15) are strongly equivalent. These two functions differ by a term of λ^2. The desired conclusion follows from a theorem (see, e.g., Theorem 2.2 in Golubitsky and Schaeffer 1985) which states that given two smooth functions g and p, if

$$T(g+p) = T(g),$$

then $g+p$ is strongly equivalent to g. Here $T(g)$ is the restricted tangent space† of g, which consists of all smooth functions of the form

$$a(x,\lambda)g(x,\lambda) + [b(x,\lambda)x + c(x,\lambda)\lambda]g_x(x,\lambda), \tag{3.17}$$

where $a(x,\lambda), b(x,\lambda)$ and $c(x,\lambda)$ are smooth functions. The theorem can be proved (Golubitsky and Schaeffer 1985, II. 11), by construction of appropriate S and X in the equivalence transformation and solving certain ordinary differential equations. With the theorem, it now suffices to show that

$$T(g) = T(f).$$

We shall give the proof of $T(f) \subset T(g)$. The proof of the reverse containment is similar. By the definition of the restricted tangent space, a function in $T(g)$ has the form of (3.17). Using (3.14) and (3.15), we find that

$$\begin{aligned} ag + (bx + c\lambda)g_x &= a(f + a_6\lambda^2) + (bx + c\lambda)(f_x + a_{6,x}\lambda^2) \\ &= \tilde{a}f + (\tilde{b}x + \tilde{c}\lambda)f_x \end{aligned} \tag{3.18}$$

where $\tilde{a}, \tilde{b}$ and $\tilde{c}$ are smooth functions given by

$$\tilde{a} \equiv a - \frac{[aa_6 + (bx + c\lambda)a_{6,x}](3a_1 + a_{1,x}x)^2 x}{[2a_1a_5 + (a_5a_{1,x} - a_1a_{5,x})x](a_5 + a_{5,x}x)},$$

$$\tilde{b} \equiv b + \frac{a_1[aa_6 + (bx + c\lambda)a_{6,x}](3a_1 + a_{1,x}x)x}{[2a_1a_5 + (a_5a_{1,x} - a_1a_{5,x})x](a_5 + a_{5,x}x)},$$

$$\tilde{c} \equiv c + \frac{aa_6 + (bx + c\lambda)a_{6,x}}{a_5 + a_{5,x}x}.$$

The last expression in (3.18) is a function in $T(f)$.

Although the above example of the pitchfork bifurcation is one of the simplest, it does demonstrate some preliminary yet essential ideas used for solving a recognition problem. A full exposition of these ideas requires sophisticated

† The definition of general tangent space pertains to the general equivalence.

mathematical treatment. Here we offer only a brief discussion of these ideas in elementary language.

The Taylor series expansion of a smooth function g consists of a collection of monomials, which can be divided into three classes: low-, intermediate-, and higher-order terms. We shall discuss in sequence the treatment of these terms in solving a recognition problem.

The low-order terms are those monomials $x^k\lambda^l$ such that $\partial^{k+l}g/\partial x^k\partial\lambda^l = 0$ in the defining conditions. In the case of (3.10), they are $1, x, \lambda$ and x^2 terms. These terms do not appear in the Taylor expansion, as evident in (3.12), and hence need no treatment in the equivalence transformation. The exclusion of the low-order terms amounts to identifying the smallest intrinsic ideal containing the given function g. In the terminology of algebra, an *intrinsic ideal* is a linear space which is closed under multiplication by smooth functions, and which is invariant under strong equivalence transformation.

The higher-order terms are those monomials that can be transformed away through the strong equivalence transformation. In the case of (3.10), they are $\lambda^2, \lambda x^2, \lambda^2 x, \lambda^3, x^4, ...$ These terms can be determined by using the theorem stated above and by examining the tangent spaces of a normal form and its perturbations, as we did for the λ^2 term above. Sometimes, these terms can be more simply identified by the operation of absorbing them in the Taylor expansion through proper redefinition of the coefficients, as we did for the $\lambda x^2, \lambda^2 x$ and λ^3 terms when getting from (3.12) to (3.14). The higher-order terms are endowed with certain algebraic structures. In particular, they form an intrinsic ideal which can be generated by a finite number of functions.

It is worth noting that it is in general incorrect to identify the higher-order terms as those monomials $x^k\lambda^l$ such that $\partial^{k+l}g/\partial x^k\partial\lambda^l$ do not appear in the defining conditions and the nondegeneracy conditions. Otherwise, the solution of a recognition problem would be all too simple. While this happens to be the case for the pitchfork bifurcation problem, as well as for some other recognition problems, there are examples for which it is not true. For instance, the defining conditions and the nondegeneracy conditions for the normal form $x^2 - \lambda^2$ are

$$g = g_x = g_\lambda = 0,\ g_{xx} > 0,\ g_{xx}g_{\lambda\lambda} - g_{x\lambda}^2 < 0. \tag{3.19}$$

Although $g_{x\lambda}$ appears in (3.19), the $x\lambda$ term is of higher order, and can be absorbed by x^2 and λ^2 terms using an equivalence transformation.

The monomials that are neither low-order nor higher-order are intermediate-order terms. These are the terms that survive in the normal form. After reducing the given function to finite intermediate-order terms, one only needs to transform their coefficients, which are smooth functions, into constants, usually 1 or -1, to reach the final expression of the normal form. This often can be done by elementary calculations. The essence of such calculations is associated with the representation of a certain Lie group of strong equivalence transformations.

9.3.3 Solutions of some recognition problems

In Table 1 below, we list the normal form, the defining conditions and the nondegeneracy conditions for several recognition problems that are often encountered in bifurcation analysis. This serves as perhaps the most important practical data that one would need when working on bifurcation problems in elasticity. Detailed derivation of these solutions can be found in Golubitsky and Schaeffer (1985, IV. 2).

Table 1. Solution of the Recognition Problem for Several Singular Functions

Nomenclature	Normal form	Defining conditions	Nondegeneracy conditions
Limit point	$\epsilon x^2 + \delta\lambda$	$g = g_x = 0$	$\epsilon = \mathrm{sgn}(g_{xx}),\ \delta = \mathrm{sgn}(g_\lambda)$
Simple bifurcation	$\epsilon(x^2 - \lambda^2)$	$g = g_x = g_\lambda = 0$	$\epsilon = \mathrm{sgn}(g_{xx})$, $g_{xx}g_{\lambda\lambda} - g_{x\lambda}^2 < 0$
Isola center	$\epsilon(x^2 + \lambda^2)$	$g = g_x = g_\lambda = 0$	$\epsilon = \mathrm{sgn}(g_{xx})$, $g_{xx}g_{\lambda\lambda} - g_{x\lambda}^2 > 0$
Hysteresis	$\epsilon x^3 + \delta\lambda$	$g = g_x = g_{xx} = 0$	$\epsilon = \mathrm{sgn}(g_{xxx}),\ \delta = \mathrm{sgn}(g_\lambda)$
Asymmetric cusp	$\epsilon x^2 + \delta\lambda^3$	$g = g_x = g_\lambda = 0$, $g_{xx}g_{\lambda\lambda} - g_{x\lambda}^2 = 0$	$\epsilon = \mathrm{sgn}(g_{xx}),\ \delta = \mathrm{sgn}(g_{vvv})$
Pitchfork	$\epsilon x^3 + \delta\lambda x$	$g = g_x = g_\lambda = 0$, $g_{xx} = 0$	$\epsilon = \mathrm{sgn}(g_{xxx}),\ \delta = \mathrm{sgn}(g_{x\lambda})$
Quartic fold	$\epsilon x^4 + \delta\lambda$	$g = g_x = g_{xx} = 0$, $g_{xxx} = 0$	$\epsilon = \mathrm{sgn}(g_{xxxx}),\ \delta = \mathrm{sgn}(g_\lambda)$
——	$\epsilon x^2 + \delta\lambda^4$	$g = g_x = g_\lambda = 0$, $g_{xx}g_{\lambda\lambda} - g_{x\lambda}^2 = 0$, $g_{vvv} = 0$	$\epsilon = \mathrm{sgn}(g_{xx})$, $\delta = \mathrm{sgn}(g_{vvvv} - 3g_{vvx}^2/g_{xx})$
Winged cusp	$\epsilon x^3 + \delta\lambda^2$	$g = g_x = g_\lambda = 0$, $g_{xx} = g_{x\lambda} = 0$	$\epsilon = \mathrm{sgn}(g_{xxx}),\ \delta = \mathrm{sgn}(g_{\lambda\lambda})$
——	$\epsilon x^4 + \delta\lambda x$	$g = g_x = g_\lambda = 0$, $g_{xx} = g_{xxx} = 0$	$\epsilon = \mathrm{sgn}(g_{xxxx})$, $\delta = \mathrm{sgn}(g_{x\lambda})$
——	$\epsilon x^5 + \delta\lambda$	$g = g_x = g_{xx} = 0$, $g_{xxx} = g_{xxxx} = 0$	$\epsilon = \mathrm{sgn}(g_{xxxxx})$, $\delta = \mathrm{sgn}(g_\lambda)$

Here, the function g and its derivatives are again evaluated at the origin, the anticipated bifurcation point. The equation $\epsilon = \mathrm{sgn}(f)$ means that $f \neq 0$ and

$$\epsilon = \begin{cases} 1 & \text{if } f > 0 \\ -1 & \text{if } f < 0. \end{cases}$$

In the defining conditions for the asymmetric cusp singularity, the equation $g_{xx}g_{\lambda\lambda} - g_{x\lambda}^2 = 0$ means that the Hessian matrix of g, given by

$$\begin{pmatrix} g_{xx} & g_{x\lambda} \\ g_{x\lambda} & g_{\lambda\lambda} \end{pmatrix},$$

has a zero eigenvalue with eigenvector

$$v = \begin{pmatrix} a \\ 1 \end{pmatrix}, \qquad a \equiv -\frac{g_{x\lambda}}{g_{xx}}.$$

A subscript v of g denotes the derivative of g in the direction of v. One may take

$$\frac{\partial}{\partial v} = a\frac{\partial}{\partial x} + \frac{\partial}{\partial \lambda},$$

and hence have

$$g_{vvv} = a^3 g_{xxx} + 3a^2 g_{xx\lambda} + 3a g_{x\lambda\lambda} + g_{\lambda\lambda\lambda}.$$

The expressions g_{vvvv} and g_{vvx} for the normal form $\epsilon x^2 + \delta\lambda^4$ are defined similarly.

9.3.4 Universal unfolding

The theory of universal unfoldings deals with imperfect bifurcation in which the bifurcation equation is subjected to small perturbations. While perturbations can be introduced to the bifurcation equation in infinitely many ways, the qualitative behavior of all possible perturbed bifurcation diagrams can be captured by introducing a finite number of parameters in the normal form of the unperturbed bifurcation equation. A comprehensive treatment of universal unfolding theory can be found in Golubitsky and Schaeffer (1985). Here we only state the definitions of unfolding and universal unfolding, and compile the universal unfoldings of several normal forms.

Let $g(x, \lambda)$ be a smooth function defined in a neighborhood $\mathcal{N}$ of the origin in $\mathbb{R} \times \mathbb{R}$, and let $G(x, \lambda, \alpha)$ be a smooth function defined in a neighborhood of the origin in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^k$, k being a positive integer. The function G is said to be an *unfolding* of g if

$$G(x, \lambda, 0) = g(x, \lambda) \quad \forall (x, \lambda) \in \mathcal{N}.$$

For example, the function

$$G(x, \lambda, \alpha) = \lambda x - x^3 + \alpha_1 + \alpha_2 x + \alpha_3 x^2 \tag{3.20}$$

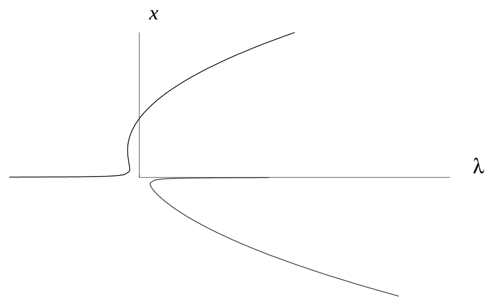

Fig. 3. Bifurcation diagram of an unfolding of the pitchfork.

is an unfolding of the function

$$g(x, \lambda) = \lambda x - x^3,$$

which is the normal form of the pitchfork bifurcation. We note that the bifurcation diagrams of these two functions can be drastically different even for small values of α. This is evident by comparing Figure 2 with Figure 3, which is the bifurcation diagram for $G(x, \lambda, \alpha) = 0$ with $\alpha_1 > 0, \alpha_2 = 0, \alpha_3 = 4\alpha_1^{1/3}$. The unperturbed equation has continuous bifurcation solutions, while the perturbed equation has discontinuous bifurcation solutions.

An unfolding G of g is *universal* if any other unfolding of g is equivalent to G and if G has the minimum number of parameters α_i. Roughly speaking, a universal unfolding $G(x, \lambda, \alpha)$ of $g(x, \lambda)$ is a function with the minimum dimension of parameter α, such that any small perturbation of g is equivalent to G for some small α. The unfolding $G(x, \lambda, \alpha)$ given by (3.20) is not universal because the term $\alpha_2 x$ is redundant. On the other hand, the function

$$G(x, \lambda, \alpha) = \lambda x - x^3 + \alpha_1 + \alpha_2 x^2$$

is a universal unfolding of the pitchfork. In general, the form of a universal unfolding is not unique. For example, another universal unfolding of the pitchfork is given by

$$G(x, \lambda, \alpha) = \lambda x - x^3 + \alpha_1 + \alpha_2 \lambda.$$

In Table 2, we list universal unfoldings of the singular functions whose normal form were listed in Table 1. The derivation of these universal unfoldings can be found in Golubitsky and Schaeffer (1985, IV. 3).

Thus far, we have discussed the recognition problem of bifurcation equations with one state variable. Bifurcation equations with multiple variables and bifurcation equations with symmetry have been treated in singularity theory. We shall not discuss the theoretical development pertaining to these issues. Instead, in the subsequent sections we shall present the analyses of a few bifurcation problems in elasticity with these features. This is not intended to be a complete exposition of the theory. We shall only introduce necessary definitions, discuss the related solution techniques, and give relevant references.

Table 2. Universal Unfoldings for Several Singular Functions

Nomenclature	Normal Form	Universal Unfolding
Limit point	$\epsilon x^2 + \delta\lambda$	$\epsilon x^2 + \delta\lambda$
Simple bifurcation	$\epsilon(x^2 - \lambda^2)$	$\epsilon(x^2 - \lambda^2 + \alpha)$
Isola center	$\epsilon(x^2 + \lambda^2)$	$\epsilon(x^2 + \lambda^2 + \alpha)$
Hysteresis	$\epsilon x^3 + \delta\lambda$	$\epsilon x^3 + \delta\lambda + \alpha x$
Asymmetric cusp	$\epsilon x^2 + \delta\lambda^3$	$\epsilon x^2 + \delta\lambda^3 + \alpha_1 + \alpha_2\lambda$
Pitchfork	$\epsilon x^3 + \delta\lambda x$	$\epsilon x^3 + \delta\lambda x + \alpha_1 + \alpha_2 x^2$
Quartic fold	$\epsilon x^4 + \delta\lambda$	$\epsilon x^4 + \delta\lambda + \alpha_1 x + \alpha_2 x^2$
——	$\epsilon x^2 + \delta\lambda^4$	$\epsilon x^2 + \delta\lambda^4 + \alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2$
Winged cusp	$\epsilon x^3 + \delta\lambda^2$	$\epsilon x^3 + \delta\lambda^2 + \alpha_1 + \alpha_2 x + \alpha_3\lambda x$
——	$\epsilon x^4 + \delta\lambda x$	$\epsilon x^4 + \delta\lambda x + \alpha_1 + \alpha_2\lambda + \alpha_3 x^2$
——	$\epsilon x^5 + \delta\lambda$	$\epsilon x^5 + \delta\lambda + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$

9.4 Bifurcation of pure homogeneous deformations with Z_2 symmetry

In this section, we present a bifurcation analysis for pure homogeneous deformations of a homogeneous, isotropic, incompressible elastic body under dead load tractions with Z_2 symmetry. Physically, this is the situation of an elastic sheet being stretched on its four edges by two pairs of perpendicular, uniformly distributed forces of equal magnitude. This is perhaps the simplest bifurcation problem with two state variables and a symmetry structure. In particular, we are interested in symmetry breaking bifurcation solutions. The existence of asymmetric equilibrium deformations under symmetric loads is suggested by the experiment of Treloar (1948), and has been studied by a number of authors, including Ogden (1985), Kearsley (1986), Chen (1991), and MacSithigh and Chen (1992a, b).

The equations of equilibrium are (3.6) in Chapter 1 and repeated here as

$$\hat{W}_1(\lambda_1, \lambda_2) = t, \; \hat{W}_2(\lambda_1, \lambda_2) = t, \tag{4.1}$$

where λ_1 and λ_2 are two principal stretches of a pure homogeneous deformation, $\hat{W}(\lambda_1, \lambda_2)$ is the reduced strain-energy function for isotropic, incompressible materials, as defined in (3.4) of Chapter 1, and the subscript i denotes the partial derivative with respect to λ_i. Also, for equi-biaxial stretch, we have set the two principal Biot stresses to t. The strain-energy function $\hat{W}$ is assumed to be C^∞.

By the isotropy, the strain-energy function has the symmetry

$$\hat{W}(\lambda_1, \lambda_2) = \hat{W}(\lambda_2, \lambda_1). \tag{4.2}$$

This symmetry, as well as the symmetry of the load, is passed to the bifurcation equations (4.1), which are hence invariant under the action of the interchange permutation group. Now we explore such a symmetry.

Let $\lambda_1 = \lambda_2 = \lambda$ be a symmetric equilibrium solution of (4.1) for $t = T$. We make the change of variables

$$x = \frac{\lambda_1 + \lambda_2}{2} - \lambda, \; y = \frac{\lambda_1 - \lambda_2}{2}, \; \tau = t - T.$$

It is observed that y is a measure of the departure of a homogeneous deformation from the symmetric deformation. The equations of equilibrium (4.1) can be written in an equivalent form

$$\hat{W}_1(\lambda + x + y, \lambda + x - y) + \hat{W}_2(\lambda + x + y, \lambda + x - y) - 2(T + \tau) = 0, \tag{4.3}$$

$$\hat{W}_1(\lambda + x + y, \lambda + x - y) - \hat{W}_2(\lambda + x + y, \lambda + x - y) = 0. \tag{4.4}$$

It follows from (4.2) that the left-hand side of (4.3) is even in y. By a theorem due to Whitney (1943), there exists a smooth function p, such that

$$p(x, y^2, \tau) = \hat{W}_1(\lambda + x + y, \lambda + x - y) + \hat{W}_2(\lambda + x + y, \lambda + x - y) - 2(T + \tau). \tag{4.5}$$

Similarly, the left-hand side of (4.4) is an odd function of y, and there exists a smooth function q, such that

$$yq(x, y^2) = \hat{W}_1(\lambda + x + y, \lambda + x - y) - \hat{W}_2(\lambda + x + y, \lambda + x - y). \tag{4.6}$$

Now the equations of equilibrium (4.3) and (4.4) can be written as

$$g(x, y, \tau) = 0, \tag{4.7}$$

where $g : \mathcal{N} \to \mathbb{R}^2$ is given by

$$g(x, y, \tau) \equiv (p(x, y^2, \tau), yq(x, y^2)), \tag{4.8}$$

$\mathcal{N}$ being a neighborhood of $\mathbb{R}^2 \times \mathbb{R}$.

The symmetry property of the function g can be described by the Z_2 symmetry group that has the following linear representation on $\mathbb{R}^2$:

$$Z_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

In general terms, a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be Z_2-*equivariant* if

$$f(\gamma z) = \gamma f(z) \qquad \forall \gamma \in Z_2,\ z \in \mathbb{R}^2.$$

By our choice of λ and T, the origin is a solution of (4.7). Henceforth, we omit the arguments of p, q and g and their derivatives when they are evaluated at $x = y = \tau = 0$. It follows from the discussion in Section 9.2.1 that a necessary condition for the origin to be a bifurcation point is that the Fréchet derivative of g with respect to the state variable (x, y) be not invertible, i.e.,

$$p_1 q = 0.$$

Here and henceforth, a subscript i of p or q denotes the partial derivative with respect to the ith argument.

We now introduce the definition of strong equivalence of Z_2-equivariant functions. Two smooth Z_2-equivariant functions $g, h : \mathcal{N} \to \mathbb{R}^2$ are said to be *strongly* Z_2-*equivalent* if there exist smooth functions $S : \mathcal{N} \to \mathbb{M}^{2\times 2}$ and $Z : \mathcal{N} \to \mathbb{R}^2$, such that

$$Z(0,0) = 0,\ \det S > 0,\ \mathrm{tr} S > 0,\ \det \nabla Z > 0,\ \mathrm{tr} \nabla Z > 0, \tag{4.9}$$

$$S(\gamma z, \tau)\gamma = \gamma S(z, \tau),\ Z(\gamma z, \tau) = \gamma Z(z, \tau) \qquad \forall \gamma \in Z_2,\ (z, \tau) \in \mathcal{N}, \tag{4.10}$$

and that

$$g(z, \tau) = S(z, \tau) h(Z(z, \tau), \tau). \tag{4.11}$$

An immediate consequence of the above definition is that strong Z_2-equivalence preserves the solution set and its symmetry near the origin.

Recognition problems for strong Z_2-equivalence can be solved by using the techniques discussed in Section 9.3. Here we list, in Table 3, the normal form, defining conditions and nondegeneracy conditions of several recognition problems. Detailed derivation of the last three solutions can be found in Golubitsky, Stewart and Shaeffer (1988, XIX. 2). Below we give a direct verification of the second solution, which corresponds to a pitchfork bifurcation in two state variables. The verification of the first solution is similar.

We first assume that the right-hand side of (4.8) is strongly Z_2-equivalent to $(\epsilon_1 x, \epsilon_2 y^3 + \epsilon_3 \tau y)$. Then there exist transformation functions S and Z that satisfy (4.9) and (4.10). By (4.10) and Whitney's theorem (1943), S and Z must have the forms

$$S = \begin{pmatrix} S_{11}(x, y^2, \tau) & yS_{12}(x, y^2, \tau) \\ yS_{21}(x, y^2, \tau) & S_{22}(x, y^2, \tau) \end{pmatrix}, \; Z = \begin{pmatrix} Z_1(x, y^2, \tau) \\ yZ_2(x, y^2, \tau) \end{pmatrix},$$

where S_{ij} and Z_i are smooth functions. By (4.9), these functions satisfy

$$Z_1(0,0,0) = 0, \; S_{11} > 0, \; S_{22} > 0, \; Z_{1,1} > 0, \; Z_2 > 0. \tag{4.12}$$

Equation (4.11) now reads

$$\begin{pmatrix} p \\ yq \end{pmatrix} = \begin{pmatrix} S_{11} & yS_{12} \\ yS_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \epsilon_1 Z_1 \\ \epsilon_2 y^3 Z_2^3 + \epsilon_3 \tau y Z_2 \end{pmatrix}. \tag{4.13}$$

Solving (4.13) for p and q, taking their derivatives and evaluating at the origin with the aid of (4.12), we find the listed defining conditions and nondegeneracy conditions.

Table 3. Solution of the Recognition Problem for Several Z_2-equivariant Singular Functions

Normal Form	Defining Conditions	Nondegeneracy Conditions
$(\epsilon_1 x^2 + \epsilon_2 \tau, \epsilon_3 y)$	$p = p_1 = 0$	$\epsilon_1 = \mathrm{sgn}(p_{11}),\ \epsilon_2 = \mathrm{sgn}(p_3),$ $\epsilon_3 = \mathrm{sgn}(q)$
$(\epsilon_1 x, \epsilon_2 y^3 + \epsilon_3 \tau y)$	$p = q = 0$	$\epsilon_1 = \mathrm{sgn}(p_1),$ $\epsilon_2 = \mathrm{sgn}(q_2 - p_2 q_1/p_1),$ $\epsilon_3 = \mathrm{sgn}(-p_3 q_1/p_1)$
$(\epsilon_1 x^2 + \epsilon_2 y^2 + \epsilon_3 \tau, \epsilon_4 xy)$	$p = q = p_1 = 0$	$\epsilon_1 = \mathrm{sgn}(p_{11}),\ \epsilon_2 = \mathrm{sgn}(p_2),$ $\epsilon_3 = \mathrm{sgn}(p_3),\ \epsilon_4 = \mathrm{sgn}(q_1)$
$(\epsilon_1 x^3 + \epsilon_2 y^2 + \epsilon_3 \tau, \epsilon_4 xy)$	$p = q = p_1 = p_{11} = 0$	$\epsilon_1 = \mathrm{sgn}(p_{111}),\ \epsilon_2 = \mathrm{sgn}(p_2),$ $\epsilon_3 = \mathrm{sgn}(p_3),\ \epsilon_4 = \mathrm{sgn}(q_1)$
$(\epsilon_1 x^2 + \epsilon_2 y^4 + \epsilon_3 \tau, \epsilon_4 xy)$	$p = q = p_1 = p_2 = 0$	$\epsilon_1 = \mathrm{sgn}(p_{11}),$ $\epsilon_2 = \mathrm{sgn}(p_{11} q_2^2 - 2p_{12} q_1 q_2$ $+ p_{22} q_1^2),$ $\epsilon_3 = \mathrm{sgn}(p_3),\ \epsilon_4 = \mathrm{sgn}(q_1)$

Next, let the functions $p(x, y^2, \tau)$ and $q(x, y^2)$ be given that satisfy the given defining conditions and the nondegeneracy conditions. Then they can be written, by Taylor's theorem, as

$$p = p_1 x + p_2 y^2 + p_3 \tau, \; q = q_1 x + q_2 y^2,$$

where p_i and q_i are smooth functions of x, y^2 and τ, the notation being so chosen that the values of these functions at the origin are identical to those appearing in the nondegeneracy conditions. Now we choose

$$S \equiv \begin{pmatrix} 1 & 0 \\ y\frac{q_1}{p_1} & \frac{|p_3 q_1|^{3/2}}{|p_1||p_1 q_2 - p_2 q_1|^{1/2}} \end{pmatrix}, \qquad Z \equiv \begin{pmatrix} \epsilon_1 p \\ y\frac{|p_1 q_2 - p_2 q_1|^{1/2}}{|p_3 q_1|^{1/2}} \end{pmatrix},$$

which satisfy conditions (4.9) and (4.10). Furthermore, a straightforward calculation shows that

$$\begin{pmatrix} p \\ yq \end{pmatrix} = S \begin{pmatrix} \epsilon_1(\epsilon_1 p) \\ \epsilon_2 (y\frac{|p_1 q_2 - p_2 q_1|^{1/2}}{|p_3 q_1|^{1/2}})^3 + \epsilon_3 \tau y \frac{|p_1 q_2 - p_2 q_1|^{1/2}}{|p_3 q_1|^{1/2}} \end{pmatrix}.$$

This establishes the desired equivalence.

As a particular example, we consider the strain-energy function of the Mooney-Rivlin form

$$\hat{W}(\lambda_1, \lambda_2) = C_1(\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2 \lambda_2^2} - 3) + C_2(\lambda_1^2 \lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} - 3), \tag{4.14}$$

where C_1 and C_2 are nonnegative constants and are not both zero. When $C_2 = 0$, the strain-energy function is said to be of neo-Hookean form. A symmetric solution of (4.1) is given by

$$\lambda_1 = \lambda_2 = \lambda, \ t = T \equiv \frac{2(C_1 + C_2\lambda^2)(\lambda^6 - 1)}{\lambda^5}. \tag{4.15}$$

It is obvious that T is monotone increasing in λ.

Substitution of (4.14) and (4.15) into (4.5) and (4.6) leads to functions $p(x, y^2, \tau)$ and $q(x, y^2, \tau)$. Evaluating these functions and their derivatives at the origin, we find that

$$p = 0, \ q = 4C_1(1 + \frac{1}{\lambda^6}) + 4C_2(-\lambda^2 + \frac{3}{\lambda^4}),$$

$$p_1 = 4C_1(1 + \frac{5}{\lambda^6}) + 12C_2(\lambda^2 + \frac{1}{\lambda^4}), \ p_2 = -12\frac{C_1}{\lambda^7} - 4C_2(\lambda + \frac{6}{\lambda^5}), \ p_3 = -2.$$

$$q_1 = -24\frac{C_1}{\lambda^7} - 8C_2(\lambda + \frac{6}{\lambda^5}), \ q_2 = 12\frac{C_1}{\lambda^8} + 4C_2(1 + \frac{10}{\lambda^6}).$$

It is observed that $p_1 > 0$. Therefore, of the normal forms in Table 3, only the second one is possible. When $C_2 = 0, q$ is always positive. This means that no bifurcation exists for neo-Hookean materials. It is further observed that q is monotone decreasing in λ, and that when $C_2 > 0, q$ becomes negative for sufficiently large λ. Therefore, there is a unique λ for which $q = 0$. This corresponds to a pitchfork bifurcation point. The coefficients in the normal form are readily calculated. It is found that $\epsilon_1 = 1, \epsilon_3 = -1$, and $\epsilon_2 = 1$ when

$C_1/C_2 > 0.08731$ and $\epsilon_2 = -1$ when $C_1/C_2 < 0.08731$. In the former case, equations (4.3) and (4.4) are strongly equivalent to

$$x = 0,\ y^3 - \tau y = 0.$$

That is, when t is less than T, there is only the symmetric solution, and as t increases from T, there exists one symmetric solution branch and two asymmetric solution branches. It has been shown by Chen (1991) with use of an energy stability criterion that the symmetric solution branch ceases to be stable at the bifurcation point, while the two asymmetric solution branches are stable.

9.5 Bifurcation of pure homogeneous deformations with D_3 symmetry

The problem to be discussed in this section is similar, in spirit, to that in the previous section. Again, we consider the bifurcation of pure homogeneous deformations of a homogeneous, isotropic, incompressible elastic body under dead load tractions. The difference is that the loads considered in this section have a D_3 symmetry. As a result, the characteristics of bifurcation for the present problem are quite different from those for the previous problem with Z_2 symmetry. Among other things, the present problem demonstrates the utility of universal unfolding, which reveals the existence of a secondary bifurcation from the solution branch with Z_2 symmetry to solution branches with no symmetry, while the primary bifurcation is from the D_3 symmetric solutions to the Z_2 symmetric solutions. Physically, this analysis describes pure homogeneous deformations of an elastic cube being stretched on its six faces by three pairs of perpendicular, uniformly distributed forces of equal magnitude. This problem was first studied by Rivlin (1948, 1974). The analysis presented here was given by Ball and Schaeffer (1983).

The equations of equilibrium are (2.96) in Chapter 1 and repeated here as

$$t = W_i(\lambda_1, \lambda_2, \lambda_3) - p\lambda_i^{-1} \qquad i = 1, 2, 3, \tag{5.1}$$

where t is the magnitude of the principal Biot stresses, p the hydrostatic pressure required by the incompressibility constraint, and λ_1, λ_2 and λ_3 are again the principal stretches, which satisfy the incompressibility constraint

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{5.2}$$

We eliminate p from (5.1) to obtain

$$\lambda_1(W_1 - t) - \lambda_2(W_2 - t) = 0, \quad \lambda_1(W_1 - t) - \lambda_3(W_3 - t) = 0, \tag{5.3}$$

By the isotropy, the strain-energy function is symmetric, i.e.

$$W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_2, \lambda_1, \lambda_3) = W(\lambda_1, \lambda_3, \lambda_2). \tag{5.4}$$

Endowed with this material symmetry and the symmetry of loads, the equations of equilibrium (5.1) are equivariant under the action of the interchange permutation and cyclic permutation group D_3. We now explore such a symmetry. First, it follows from (5.3) and (5.4) that $(\lambda_1, \lambda_2, \lambda_3) = (1, 1, 1)$ is an equilibrium solution for any t, reflecting the inability of an incompressible material to deform under hydrostatic pressure. Let T be the value of load at which bifurcation is anticipated. We make the change of variables

$$\lambda_1 = e^{2x},\ \lambda_2 = e^{-x+\sqrt{3}y},\ \lambda_3 = e^{-x-\sqrt{3}y},\ t = T + \tau. \tag{5.5}$$

It is observed that λ_1, λ_2 and λ_3 in (5.5) satisfy (5.2) for any values of x and y. The equations of equilibrium (5.3) can be rewritten in an equivalent form

$$g(z, \tau) = 0,$$

where $g : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is a smooth function given by

$$\begin{aligned} g(x + iy, \tau) \equiv\ & 2\lambda_1(W_1 - t) - \lambda_2(W_2 - t) - \lambda_3(W_3 - t) \\ & + i\sqrt{3}\ [\lambda_2(W_2 - t) - \lambda_3(W_3 - t)], \end{aligned} \tag{5.6}$$

where $\lambda_1, \lambda_2, \lambda_3$ and t are related to x, y and τ through (5.5). We note that the statement that g is a smooth function means only that the real and imaginary parts of g are C^∞ functions of x and y. Indeed, as a complex function, g is not in general analytic. The action of the D_3 group on a complex number z is given by

$$\{z, \bar{z}, e^{2\pi i/3}z, e^{2\pi i/3}\bar{z}, e^{-2\pi i/3}z, e^{-2\pi i/3}\bar{z}\},$$

where a superimposed bar denotes complex conjugate. It is a simple exercise to verify that g defined in (5.6) is D_3-equivariant, i.e.,

$$g(\gamma z, \tau) = \gamma g(z, \tau) \qquad \forall \gamma \in D_3, z \in \mathbb{C}.$$

It is obvious from (5.6) and (5.5) that

$$g(0, \tau) = 0, \tag{5.7}$$

confirming the earlier observation that the identity deformation is an equilibrium solution for each value of the symmetric applied load. It has been shown by Golubitsky and Schaeffer (1982) that a D_3-equivariant function g satisfying (5.7) has the representation

$$g(z, \tau) = a(|z|^2, \operatorname{Re} z^3, \tau)z + b(|z|^2, \operatorname{Re} z^3, \tau)\bar{z}^2, \tag{5.8}$$

where $a, b : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ are smooth functions.

The definition of strong equivalence of D_3-equivariant functions is the same as that of Z_2-equivariant functions stated in (4.9)–(4.11) with proper replacement of the symmetry group. In Table 4, we list two normal forms of g that are of interest to the particular elasticity problem at hand.

As an example, we once again consider the strain-energy function of Mooney-Rivlin form, now written as

$$W(\lambda_1, \lambda_2, \lambda_3) = C_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_2(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3). \tag{5.9}$$

Table 4. Solution of the Recognition Problem for Two D_3-equivariant Singular Functions

Normal Form	Defining Conditions	Nondegeneracy Conditions
$\epsilon_1 \tau z + \epsilon_2 \bar{z}^2$	$a = 0$	$\epsilon_1 = \operatorname{sgn}(a_3),\ \epsilon_2 = \operatorname{sgn}(b)$
$(\epsilon_1 \lvert z\rvert^2 + \epsilon_2 \tau) z$ $+(\epsilon_3 \lvert z\rvert^2 + \Delta \operatorname{Re} z^3)\bar{z}^2$	$a = b = 0$	$\epsilon_1 = \operatorname{sgn}(a_1),\ \epsilon_2 = \operatorname{sgn}(a_3),$ $\epsilon_3 = \operatorname{sgn}(\frac{a_3 b_1 - a_1 b_3}{a_3}),$ $\Delta = \frac{(a_1 b_2 - a_2 b_1) a_3^2}{(a_1 b_3 - b_1 a_3)^2} \operatorname{sgn}(a_1)$

Substituting (5.9) into the right-hand side of (5.6), evaluating it for (5.5), and equating the resulting expression to the right-hand side of (5.8), we obtain a complex equation which we can solve for the two real-valued functions $a(|z|^2, \operatorname{Re} z^3, \tau)$ and $b(|z|^2, \operatorname{Re} z^3, \tau)$. A tedious but straightforward computation of derivatives of these functions leads to

$$\begin{gathered} a = 24(C_1 + C_2 - T/4), \quad a_1 = 48(C_1 + C_2 - T/16), \\ a_2 = 16(C_1 - C_2 - T/32), \quad a_3 = -6, \end{gathered} \tag{5.10}$$

$$\begin{gathered} b = 24(C_1 - C_2 - T/8), \quad b_1 = 24(C_1 - C_2 - T/32), \\ b_2 = (32/5)(C_1 + C_2 - T/64), \quad b_3 = -3. \end{gathered} \tag{5.11}$$

By $(5.10)_1$, the defining condition $a = 0$ requires

$$T = 4(C_1 + C_2), \tag{5.12}$$

which gives the value of the load at the bifurcation point. Substituting (5.12) into $(5.11)_1$ shows that $b = 0$ if and only if

$$C_1 = 3C_2. \tag{5.13}$$

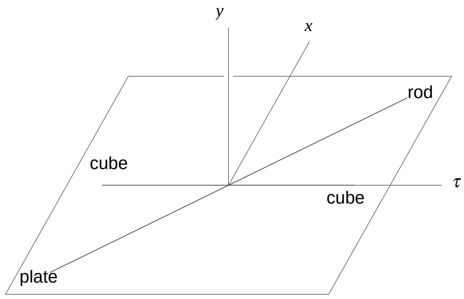

Fig. 4. Bifurcation diagram of $-\tau z + \bar{z}^2 = 0$.

We first consider the case when $C_1 > 3C_2$, which includes the neo-Hookean material as a special subcase. The normal form of the bifurcation equation in this case is

$$-\tau z + \bar{z}^2 = 0.$$

This equation has four solution branches

$$\begin{cases} x = 0 \\ y = 0, \end{cases} \qquad \begin{cases} x = \tau \\ y = 0, \end{cases} \qquad \begin{cases} x = -\frac{1}{2}\tau \\ y = \pm\sqrt{3}x. \end{cases}$$

These solution branches are sketched in Figure 4. To avoid an overcrowded figure, only the trivial solution branch $x = y = 0$ and the non-trivial solution branch $x = \tau, y = 0$ are plotted. The other two non-trivial solution branches can be obtained by rotating the plotted solution branch through angles $2\pi/3$ and $4\pi/3$ about the τ-axis. Assume that the undeformed body is a cube. It is clear from (5.5) that the trivial solution corresponds to the symmetric identity deformation with the shape of the cube remaining unchanged. It also follows from (5.5) that when $\tau < 0$, the plotted non-trivial solution corresponds to a deformation for which $\lambda_1 < 1$ and $\lambda_2 = \lambda_3 > 1$. That is, the deformed body is plate-like. On the other hand, the deformed body is rod-like with $\lambda_1 > 1$ and $\lambda_2 = \lambda_3 < 1$ when $\tau > 0$. Similar conclusions hold for the other two non-trivial solution branches.

We note that these bifurcation solutions occur for the neo-Hookean material. This shows that the present bifurcation problem features different characteristics from that of the equi-biaxial stretch discussed in the previous section, for which the neo-Hookean material does not permit bifurcation.

The bifurcation diagram when $C_1 < 3C_2$ is similar to the case where $C_1 > 3C_2$. We now turn our attention to the case where (5.13) holds, i.e. where

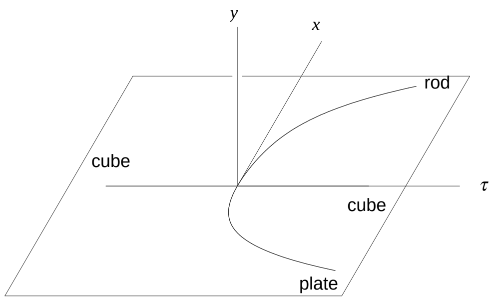

Fig. 5. Bifurcation diagram of (5.14).

$b = 0$. By (5.10) and (5.11), the coefficients of the second normal form in Table 4 are found to be

$$\epsilon_1 = 1,\ \epsilon_2 = \epsilon_3 = -1,\ \Delta = 2.$$

The normal form of the bifurcation equation is then

$$(|z|^2 - \tau)z + (-|z|^2 + 2\mathrm{Re}\, z^3)\bar{z}^2 = 0. \tag{5.14}$$

This equation again has four solution branches

$$\begin{cases} x = 0 \\ y = 0, \end{cases} \qquad \begin{cases} x^2 - x^3 + 2x^4 = \tau \\ y = 0, \end{cases} \qquad \begin{cases} 4x^2 + 8x^3 + 32x^4 = \tau \\ y = \pm\sqrt{3}x. \end{cases}$$

In Figure 5, we plot the trivial solution branch and the non-trivial solution branch that lies in the (τ, x) plane. The other two solution branches can be obtained by rotating the plotted non-trivial solution branch by $2\pi/3$ and $4\pi/3$ about the τ-axis. It is observed that the two plotted solution branches form a pitchfork bifurcation. The trivial solution again corresponds to the identity deformation of the cube. The part of the non-trivial solution branch with positive x corresponds to rod-like deformations, and the part with negative x to plate-like deformations. The other two non-trivial solution branches have similar structures.

In reality, it is unlikely that the equality (5.13) holds exactly. This may seem to diminish the significance of the solution (5.14). However, by studying the universal unfolding of (5.14), not only can one determine the bifurcation diagram when (5.13) holds approximately, it is also possible to gain information about the bifurcation solutions which is unavailable from (5.14).

A universal unfolding of (5.14) is

$$(|z|^2 - \tau)z + [-|z|^2 + (2 + \alpha_1)\mathrm{Re}\, z^3 + \alpha_2]\bar{z}^2 = 0. \tag{5.15}$$

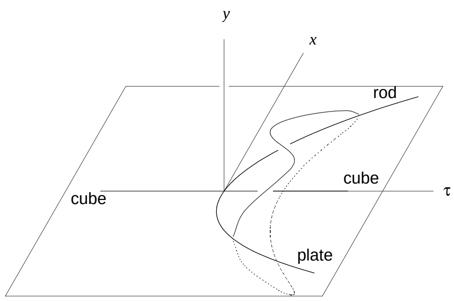

Fig. 6. Bifurcation diagram of the universal unfolding (5.15) when $\alpha_1 = 0, \alpha_2 > 0$.

This universal unfolding describes, in particular, the bifurcation diagrams for a Mooney-Rivlin material when C_1 is near $3C_2$, since we can interpret a departure of the values of material constants from those specified in (5.13) as a perturbation to the bifurcation equation (5.14). The parameter α_1 in (5.15) plays an inessential role in the bifurcation diagram of the universal unfolding. The parameter α_2, on the other hand, can change the bifurcation diagram drastically. The four solution branches in (9.5) now become

$$\begin{cases} x = 0 \\ y = 0, \end{cases} \qquad \begin{cases} \alpha_2 x + x^2 - x^3 + (2+\alpha_1)x^4 = \tau \\ y = 0, \end{cases}$$

$$\begin{cases} -2\alpha_2 x + 4x^2 + 8x^3 + 16(2+\alpha_1)x^4 = \tau \\ y = \pm\sqrt{3}x. \end{cases}$$

A noticeable feature of the universal unfolding (5.15) is the existence of a fifth solution branch when $\alpha_2 > 0$, given by

$$\begin{cases} -x^2 - y^2 + (2+\alpha_1)(x^3 - 3xy^2) + \alpha_2 = 0 \\ x^2 + y^2 - \tau = 0. \end{cases} \tag{5.16}$$

This solution branch is plotted in Figure 6, along with the trivial solution branch and the non-trivial solution branch in the (τ, x) plane. It is observed that this solution branch is connected to the other three non-trivial solution branches. Therefore, it represents a secondary bifurcation. It is further observed that the solution (5.16) possesses no symmetry, i.e. the three principal stretches are distinct.

It is worth noting that the analysis of universal unfolding is still local. The perturbation from the original normal form must be sufficiently small. In the context of the present example, this is interpreted as $C_1 - 3C_2$ being sufficiently small. However, this statement is not quantified here. For individual problems, the conclusions of such local analyses may well be valid for perturbations that are not considered to be small in a practical measure. This is because the qualitative behavior of the solution could remain unchanged when the perturbation parameters vary in a certain range. From a practical point of view, the special features predicted by a local analysis may serve as indications of the global behavior of bifurcation solutions.

9.6 Bifurcation of inflation of spherical membranes

In the last section of this chapter, we present the bifurcation analysis of an infinite-dimensional problem. We consider axisymmetric deformations of a spherical membrane inflated by an enclosed controlled mass of gas. The equations of equilibrium are a pair of ordinary differential equations. By the Liapunov-Schmidt reduction, a reduced bifurcation equation with one state variable is derived. Two normal forms of the singular function are studied. One corresponds to the pitchfork in which two non-spherical solution branches bifurcate from the spherical solution branch. The other normal form corresponds to an isola. The universal unfolding of the latter normal form suggests a bifurcation diagram in which two non-spherical solution branches bifurcate from the spherical solution branch, and then, as the amount of the gas increases further, return to the spherical solution branch. This behavior agrees with the experimental observation reported by Alexander (1971), as well as the numerical analysis of Haughton (1980) for an Ogden material. The analysis of this section is developed from the work of Chen and Healey (1991).

The theory of elastic membranes is discussed in Chapter 7 of this volume. We consider an initially spherical elastic membrane of unit radius. In spherical coordinates, an axisymmetric deformation of the membrane is represented by

$$r = r(\Theta),\ \theta = \theta(\Theta),\ \phi = \Phi.$$

The functions $r(\Theta)$ and $\theta(\Theta)$ satisfy the boundary conditions

$$r(0) = r(\pi) = 0,\ \theta(0) = 0,\ \theta(\pi) = \pi,\ \theta(\pi/2) = \pi/2. \tag{6.1}$$

Equation $(6.1)_5$ is imposed to eliminate rigid-body translations. The principal stretches of the deformed membrane are given by

$$\lambda_1 = \frac{r \sin\theta}{\sin\Theta}, \quad \lambda_2 = \sqrt{r'^2 + r^2\theta'^2}, \tag{6.2}$$

where a prime denotes the derivative with respect to Θ.

The membrane is homogeneous and isotropic, and is associated with a strain-energy function $W(\lambda_1, \lambda_2)$ which is symmetric in its two arguments. The membrane is inflated by an ideal gas, of which the mass m is taken as the control parameter. The equations of equilibrium are given by

$$p(r\sin\theta)^2\lambda_2 + 2(r\cos\theta)'W_2\sin\Theta = 0,\ (r\sin\theta)'W_1 - \lambda_2(W_2\sin\Theta)' = 0, \tag{6.3}$$

where a subscript i on W again denotes the derivative with respect to λ_i, and p is the pressure of the gas, given by

$$p = \frac{km}{V}, \tag{6.4}$$

k being a positive gas constant, and V the volume of the gas enclosed by the deformed membrane, given by

$$V = -\pi\int_0^\pi (r\sin\theta)^2(r\cos\theta)'d\Theta. \tag{6.5}$$

A spherical deformation is given by

$$r(\Theta) = \lambda,\ \theta(\Theta) = \Theta, \tag{6.6}$$

where λ is the principal stretch of the deformed membrane. The equations of equilibrium for the spherical deformation become

$$m = \frac{8\pi\lambda W_1}{3k}. \tag{6.7}$$

Here and henceforth, the derivatives of W are evaluated at $(\lambda_1, \lambda_2) = (\lambda, \lambda)$, unless otherwise stated.

A bifurcation solution at the spherical deformation (6.6) exists only if the Fréchet derivative of the nonlinear differential operator associated with (6.3) is not invertible at (6.6). This leads to the linearized equations of equilibrium

$$W_1(u_1'\cot\Theta + 2u_1 + u_2\cot\Theta) - \lambda W_{11}(u_1 + u_2') - \lambda W_{12}(u_1 + u_2\cot\Theta)$$
$$-\frac{3}{2}W_1\int_0^\pi u_1\sin\Theta d\Theta = 0, \tag{6.8}$$

$$(W_1 - \lambda W_{12})(u_1' - u_2) - \lambda W_{11}(u_2'' + u_1' + u_2'\cot\Theta - u_2\cot^2\Theta) = 0, \tag{6.9}$$

where $u_1(\Theta)$ and $u_2(\Theta)$ correspond to the radial and transverse components of the displacement, which are related to $r(\Theta)$ and $\theta(\Theta)$ through

$$r(\Theta) = \lambda + u_1(\Theta),\ \theta(\Theta) = \Theta + \frac{u_2(\Theta)}{\lambda}. \tag{6.10}$$

The functions u_1 and u_2 satisfy the boundary conditions

$$u_1'(0) = u_1'(\pi) = u_2(0) = u_2(\pi) = u_2(\pi/2) = 0. \tag{6.11}$$

Equations (6.8) and (6.9) can be converted, through a change of variables, to an inhomogeneous Legendre equation whose solution consists of Legendre polynomials. It can be shown that the non-spherical solution of lowest order that satisfies the boundary conditions (6.11) exists when

$$W_1 - \lambda(W_{11} + W_{12}) = 0, \tag{6.12}$$

and is given by

$$u_1 = \cos\Theta,\ u_2 = 0.$$

This solution, corresponding to the so-called mode-one bifurcation, suggests a pear-shaped deformation for which the two principal stretches are monotone increasing from one pole to the other. In the remainder of this section, we shall consider bifurcation solutions from the spherical solution (6.6) with λ satisfying (6.12), the corresponding mass at the bifurcation point being given by (6.7).

We now use Liapunov-Schmidt reduction to derive an algebraic equation which is equivalent to the differential equations (6.3). The analysis below follows the procedure described in Section 9.2.2. We first define

$$\mathcal{X} \equiv \{u \in C^2([0,\pi];\mathbb{R}^2) : u_1'(0) = u_1'(\pi) = u_2(0) = u_2(\pi) = u_2(\pi/2) = 0\},$$

and then define $f : \mathcal{X} \times \mathbb{R} \to C^0([0,\pi];\mathbb{R}^2)$ by

$$\begin{aligned} f(u,\mu) \quad \equiv \quad & (k(m+\mu)(r\sin\theta)^2\lambda_2 \\ & -2\pi(r\cos\theta)'W_2(\lambda_1,\lambda_2)\sin\Theta\int_0^\pi (r\sin\theta)^2(r\cos\theta)'d\Theta, \\ & (r\sin\theta)'W_1(\lambda_1,\lambda_2) - \lambda_2[W_2(\lambda_1,\lambda_2)\sin\Theta]'), \end{aligned} \tag{6.13}$$

where m is given by (6.7), λ_1 and λ_2 by (6.2), and r and θ by (6.10) with λ again satisfying (6.12). In the definition of $f(u,\mu)$, the bifurcation point has been moved to the origin, which corresponds to the spherical deformation specified by (6.6) and (6.12). The state variable u stands for the displacement from the spherical deformation, and the bifurcation parameter μ for the mass increment from that required by the spherical deformation.

The Fréchet derivative L of f with respect to u at $(u,\mu) = (0,0)$ is given by

$$\begin{aligned} Lu \quad = \quad & (\frac{4}{3}\pi\lambda^3\sin^2\Theta[W_1(2u_1 + 2u_1'\cot\Theta - 3\int_0^\pi u_1\sin\Theta d\Theta) \\ & +2\lambda W_{11}(u_2\cot\Theta - u_2')], \\ & -\lambda W_{11}\sin\Theta(u_2 - u_2\cot^2\Theta + u_2'\cot\Theta + u_2'')) \end{aligned}$$

It has been shown that the linear operator L is Fredholm of index 0 with one-dimensional kernel and co-kernel (complement of range L) spanned, respectively, by

$$e(\Theta) = (\cos\Theta, 0),\ e^*(\Theta) = (0, \sin\Theta).$$

Let P be the orthogonal projection on range L, given by

$$Py(\Theta) = y(\Theta) - [\frac{2}{\pi}\int_0^\pi e^*(t)\cdot y(t)dt]e^*(\Theta).$$

Equations (6.3) are now written as the equivalent equations

$$Pf(ze(\Theta) + w(\Theta), \mu) = 0, \tag{6.14}$$

$$\int_0^\pi e^*(\Theta)\cdot f(ze(\Theta) + w(\Theta), \mu)d\Theta = 0, \tag{6.15}$$

where $z \in \mathbb{R}$ and $w : [0,\pi] \to \mathbb{R}^2$ is orthogonal to e, i.e.

$$\int_0^\pi e(\Theta)\cdot w(\Theta)d\Theta = 0.$$

In (6.14) and (6.15), the scalar variable z is the component of the displacement $u(\Theta)$ in the direction of ker L, and $w(\Theta)$ is the remaining displacement.

The Fréchet derivative of the left-hand side of (6.14) with respect to w is an invertible linear mapping from the complement of ker L onto range L. By the implicit function theorem, one can solve equation (6.14) locally for w, and write

$$w = W(z, \mu). \tag{6.16}$$

Substituting (6.16) into (6.15) yields the reduced bifurcation equation

$$g(z,\mu) \equiv \int_0^\pi e^*\cdot f(ze + W(z,\mu), \mu)d\Theta = 0.$$

The derivatives of $g(z,\mu)$ can be found by following the analysis leading to (2.19)–(2.24). Note that here ker L and $\mathcal{R}$ have dimension one and that we have used z for the state variable and μ for the bifurcation parameter in the reduced bifurcation equation. Using (2.19) and (2.21)–(2.23), along with a lengthy but routine calculation, we find that

$$g_z = 0,\ g_\mu = (e^*, f_\mu) = 0,\ g_{zz} = (e^*, f_{uu}ee) = 0, \tag{6.17}$$

$$g_{z\mu} = (e^*, -f_{uu}e(L^{-1}Pf_\mu) + f_{u\mu}e) = \frac{k\lambda(W_{111} + 3W_{112})}{4\pi W_1}. \tag{6.18}$$

We note that f defined in (6.13) has a symmetry:

$$f(\gamma u(\pi-\Theta),\mu)=\gamma f(u(\Theta),\mu), \qquad \gamma\equiv\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

This symmetry is subsequently inherited by the functions $W(z,\mu)$ and $g(z,\mu)$. In particular, g is odd in z:

$$g(-z,\mu)=-g(z,\mu).$$

Such a function is said to have Z_2-symmetry†. As a result of Z_2-symmetry, the value of g, its derivatives with respect to μ, and its even-order derivatives with respect to z all vanish at the origin, in agreement with, for example, (6.17).

The solutions of several recognition problems for Z_2-symmetric functions are given in Golubitsky and Schaeffer (1985, VI. 5). In Table 5, we list two of them which are of interest to the particular elasticity problem at hand. The first normal form is a pitchfork, while the second normal form consists of the trivial solution branch and either a simple bifurcation or an isola center.

Table 5. Solution of the Recognition Problem for Two Z_2-symmetric Singular Functions

Normal Form	Defining Conditions	Nondegeneracy Conditions
$\epsilon z^3+\delta\mu z$	$g_z=0$	$\epsilon=\operatorname{sgn}(g_{zzz}),\ \delta=\operatorname{sgn}(g_{z\mu})$
$\epsilon z^3+\delta\mu^2 z$	$g_z=g_{z\mu}=0$	$\epsilon=\operatorname{sgn}(g_{zzz}),\ \delta=\operatorname{sgn}(g_{z\mu\mu})$

Once again we turn to specific strain-energy functions. First we examine the strain-energy function of Mooney-Rivlin form given by (4.14). A simple calculation shows that

$$W_1-\lambda(W_{11}+W_{12})=-\frac{4[3C_1+C_2(2\lambda^2+\lambda^8)]}{\lambda^5}<0.$$

That is, the Mooney-Rivlin material does not permit mode-one bifurcation. Since the deformations corresponding to such a bifurcation have been observed in experiment with spherical neoprene balloons, this result serves as an indication of the inadequacy of the Mooney-Rivlin model for rubber-like materials at large deformation.

A more accurate form of strain-energy function for a rubber-like material,

† Z_2-symmetry for functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ is discussed in Section 9.4.

proposed by Ogden (1972), is given by

$$W(\lambda_1, \lambda_2) = \sum_{n=1}^{3} \frac{\mu_n}{\alpha_n}[\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \frac{1}{\lambda_1^{\alpha_n}\lambda_2^{\alpha_n}} - 3], \tag{6.19}$$

where the dimensionless material parameters are given by

$$\mu_1 = 1.491,\ \mu_2 = 0.003,\ \mu_3 = -0.024,\ \alpha_1 = 1.3,\ \alpha_2 = 5.0,\ \alpha_3 = -2.0. \tag{6.20}$$

Substituting (6.19) and (6.20) into (6.12), we find that there are two mode-one solutions at

$$\lambda^{(1)} = 1.778, \quad \lambda^{(2)} = 2.514.$$

By (6.7), the corresponding masses of gas at the bifurcation points are

$$m^{(1)} = \frac{25.99}{k}, \quad m^{(2)} = \frac{50.79}{k}.$$

Furthermore, by (6.18), the coefficients of the first normal form in Table 5 are found to be

$$g_{zzz}^{(1)} > 0, \quad g_{zzz}^{(2)} > 0, \quad g_{z\mu}^{(1)} < 0, \quad g_{z\mu}^{(2)} > 0. \tag{6.21}$$

Hence, these two solutions correspond to pitchfork bifurcation. By the signs of the coefficients in (6.21), these two pitchforks are found to be facing each other, as shown in Figure 7.

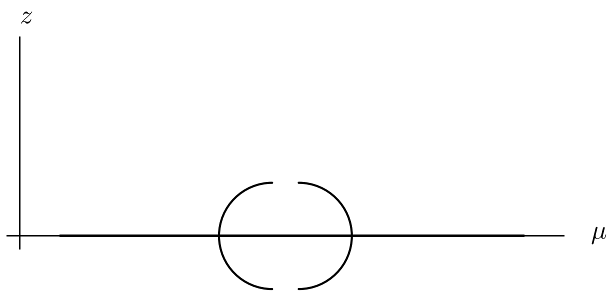

Figure 7. Bifurcation diagram with two pitchforks.

Using numerical analysis, Haughton (1980) has shown that these two pitchfork solution branches are actually connected to each other, forming a closed loop. The first normal form itself does not yield this result, since the analysis is local. However, this result can be confirmed by a study of the universal unfolding of the second normal form. Indeed, by adjusting the values of the material parameters in (6.20), it is possible to make the two bifurcation points coalesce with $g_{z\mu} = 0$ at the new bifurcation point. In the case where $g_{zzz} > 0$ and $g_{z\mu\mu} > 0$, the bifurcation equation has the second normal form in Table 5, i.e.

$$z^3 + \mu^2 z = 0. \tag{6.22}$$

A universal unfolding of (6.22) is

$$z^3 + \mu^2 z + \alpha z = 0. \tag{6.23}$$

For negative α, the bifurcation diagram of (6.23) is sketched in Figure 8, which consists of a closed loop and the trivial solution branch. This gives a qualitative description of the solutions when the values of the material parameters are near the adjusted values.

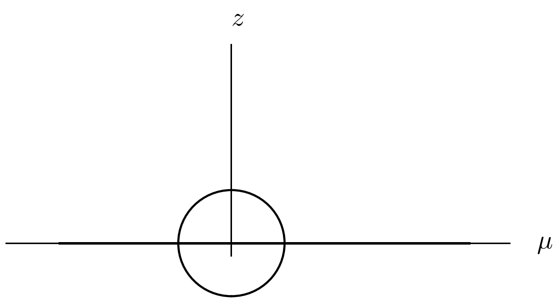

Figure 8. Bifurcation diagram of the universal unfolding (6.23) when $\alpha < 0$.

This argument is carried out alternatively in Chen and Healey (1991) for a general class of materials. The structure of the bifurcation solutions allows for a simple graphical representation. Let ξ be the reciprocal of the principal stretch of the spherical solution. Substituting (6.5)–(6.7) into (6.4), we can write the pressure as a function of ξ:

$$p(\xi) \equiv 2\xi^2 W_1(1/\xi, 1/\xi). \tag{6.24}$$

Successive differentiation of (6.24) yields

$$p'(\xi) = 4\xi W_1 - 2(W_{11} + W_{12}), \; p''(\xi) = 4W_1 - \frac{4}{\xi}(W_{11} + W_{12}) + \frac{2}{\xi^2}(W_{111} + 3W_{112}).$$

It then follows that equation (6.12) holds at a point if and only if the tangent line of the (p, ξ) curve at that point passes through the origin. Moreover, the coefficient $g_{z\mu}$ given by (6.18) vanishes at this point if and only if the curvature of the (p, ξ) curve is zero there. This give a graphical characterization of the materials which have mode-one bifurcation solutions with one of the normal

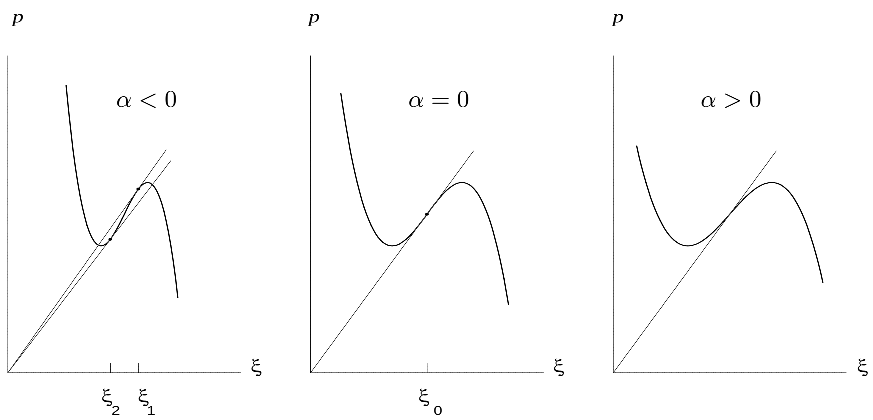

Fig. 9. Several (p, ξ) curves.

forms in Table 5. In Figure 9, we plot three (p, ξ) curves. It is observed that there are two pitchfork bifurcation points ξ_1 and ξ_2 for the material with (p, ξ) curve shown in Figure 9(a), there is one bifurcation point ξ_0 with normal form (6.22) for Figure 9(b), and no mode-one bifurcation point for Figure 9(c).

With this structure in mind, we consider a one-parameter family of strain-energy functions $W(\lambda_1, \lambda_2, \alpha)$. The above bifurcation analysis can be carried out with α being treated as a second bifurcation parameter. This leads to a reduced bifurcation equation

$$g(z, \mu, \alpha) = 0. \tag{6.25}$$

It is shown by Chen and Healey (1991) that the (p, ξ) curves of this family of materials are as shown in Figure 9 for various values of α, and that the normal form of (6.25) is precisely (6.23). We thus conclude that when α is negative and sufficiently close to zero, there are two pitchfork bifurcation solution branches that are connected to each other to form a closed loop, as shown in Figure 8.

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