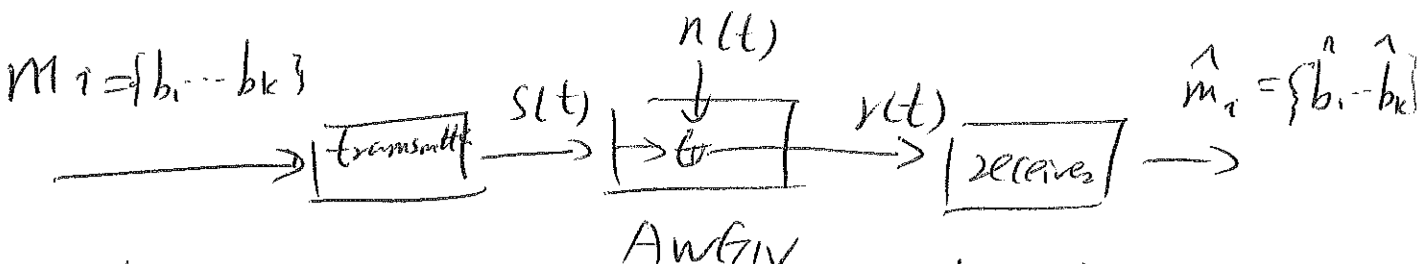
Chapter 5 Digital Modulation & Detection

5.1 Signal space analysis

Every T seconds, the system sends $k = \log_2 M$ bits. The rate is $R = k/T$. Each bit sequence of length k comprise a message $m_i = \{b_1 \cdots b_k\} \in \mathcal{M}$, where $\mathcal{M} = \{m_1 \cdots m_M\}$ is the set of all such message. ~~The~~ let The probability P_i ~~of being~~ of being the selected sequences has $\sum_{i=1}^{M} P_i = 1$

each message $m_i \in \mathcal{M}$ is mapped to a unique analog signal $s_i(t) \in S\{$ ~~Energy~~ $s_1(t), \cdots s_M(t)\}$, with energy

$$E_{s_i} = \int_0^T s_i^2(t)\,dt, \quad \forall i.$$



where $\hat{m}$ is the best estimate ~~and~~ from $r(t) = s(t) + n(t)$

The goal is the minimize the error probability

$$P_e = \sum_{i=1}^{M} P(\hat{m} \neq m_i \mid m_i \text{ sent})\, P(m_i \text{ sent})$$

5.1.2 Geometric Representation of Signals

orthogonormal basis functions $\{\phi_1(t) \cdots \phi_N(t)\}_{N \le M}$

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\, \phi_j(t), \quad 0 \le t < T$$

$$s_{ij} = \int_0^T s_i(t)\, \phi_j(t)\, dt$$

s_{ij} is projection of $s_i(t)$ onto $\phi_j(t)$

$$\int_0^T \phi_i(t)\, \phi_j(t)\, dt = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases}$$

For example $\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$$

$$s_i(t) = s_{i1} \sqrt{\frac{2}{T}} \cos(2\pi f_c t) + s_{i2} \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$$

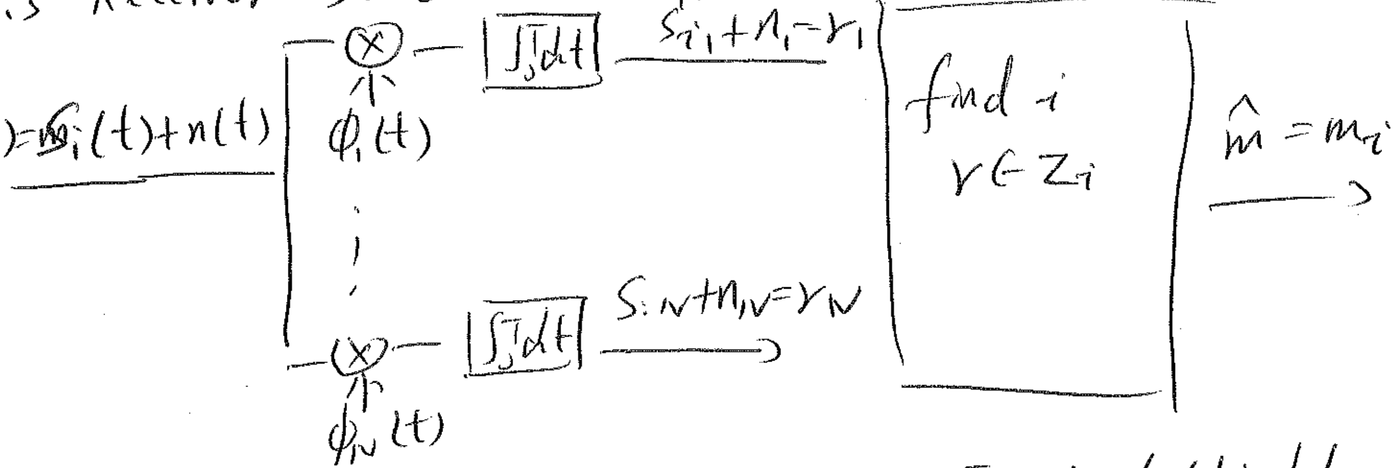
example 5.1 BPSK ~~and~~, QPSK, MQAM

- ~~Power~~ length of a vector $\|s_i\| \triangleq \sqrt{\sum_{j=1}^{N} s_{ij}}$

- distance $\|s_i - s_k\| = \sqrt{\sum_{j=1}^{N} (s_{ij} - s_{kj})^2}$

- ~~int~~ inner product $\langle s_i(t), s_k(t) \rangle = \int_0^T s_i(t) s_k(t)\, dt$

5.1.3 Receiver structure & Sufficient statistics



$$s_{ij}=\int_0^T s_i(t)\,\phi_j(t)\,dt\ ,\quad n_j=\int_0^T n(t)\,\phi_j(t)\,dt$$

$$r(t)=\sum_{j=1}^{N} r_j\,\phi_j(t)+n_r(t)$$

where $r_j=s_{ij}+n_j$ and $n_r(t)=n(t)-\sum_{j=1}^{N} n_j\,\phi_j(t)$

we want to see the mean and variance

$$\mu_{r_i|s_i}=E[r_j|s_i]=E[s_{ij}+n_j|s_{ij}]=s_{ij}$$

$$\sigma^2_{r_i|s_i}=E[r_j-\mu_{r_j}|s_i]^2=E[n_j^2|s_{ij}]=E[n_j^2]$$

$$\mathrm{cov}[r_j\, r_k|s_i]=\begin{cases}N_0/2 & j=k\\ 0 & j\neq k\end{cases}$$

This is because of AWGN.

we can write probability

$$P(r|s_i)=\prod_{j=1}^{N}P(r_j|m_i)=\frac{1}{(\pi N_0)^{N/2}}\exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}(r_j-s_{ij})^2\right]$$

we are interested in

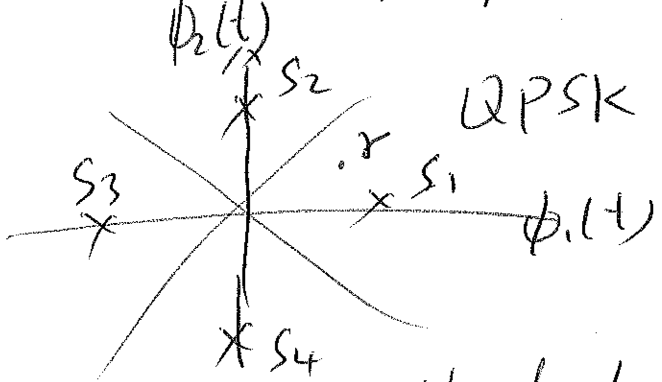
$$P(s_i|r)=\frac{P(r|s_i)\,P(s_i)}{P(r)}$$ Bayesian rule.

$\leftarrow$ sufficient statistics

5.1.4 Decision Region and maximum likelihood decision

decision region $\{Z_1 \cdots Z_M\}$

$$Z_i = \{ r : P(s_i \text{ sent} \mid r) > P(s_j \mid r),\ \forall j \neq i \}$$



Maximum likelihood decision

$$\arg\max_{s_i} P(s_i \mid r) = \arg\max_{s_i} \frac{P(r \mid s_i) P(s_i)}{P(r)} = \arg\max_{s_i} P(r \mid s_i) P(s_i)$$

if $P(s_i) = P(s_j)\ \forall i, j$

$$\arg\max_{s_i} \underbrace{P(r \mid s_i)}_{L(s_i) =},\ \forall i$$

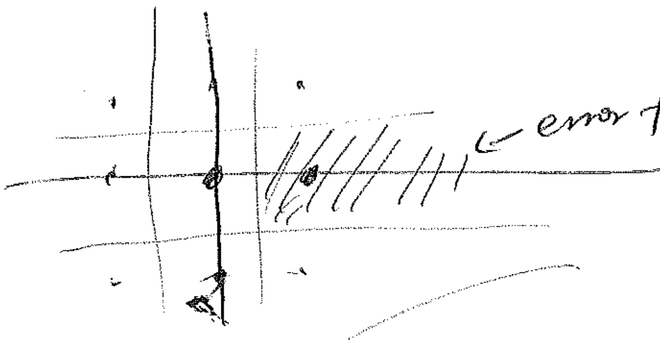
For Gaussian case

$$L(s_i) = -\frac{1}{N_0} \| r - s_i \|^2$$

$$Z_i = \{ r : \| r - s_i \| < \| r - s_j \|,\ \forall j \neq i \},\ \forall i$$

example 5.2

5.1.5 error probability & Union bound



← error free area

symbol error rate.

$$P_e(m_i \text{ sent}) = P\left(\bigcup_{\substack{k=1 \\ k\neq i}}^{M} A_{ik}\right) \leq \sum_{\substack{k=1 \\ k\neq i}}^{M} P(A_{ik})$$

$$P(A_{ik}) = P(\|s_k - r\| < \|s_i - r\| \mid s_i \text{ sent})$$

$$= P(\|s_k - (s_i + n)\| < \|s_i - (s_i + n)\|)$$

$$= P(\|n + s_i - s_k\| < \|n\|)$$

$$= P\left(n > \frac{d_{ik}}{2}\right) = \int_{d_{ik}/2}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left(\frac{-v^2}{N_0}\right) dv$$

BPSK

$$= Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

$$P_e \leq (M-1)\, Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

example 5.3

Gray coding $\quad P_b \simeq \dfrac{P_e}{\log_2 M}$